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Duality between Higher Spin Gravity in (Anti-)de Sitter Spacetime and Conformal Field Theories

Masterarbeit – Master's Thesis

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1 Introduction

The AdS/CFT correspondence by Juan Maldacena is one of the most important conceptional developments in theoretical physics in the last two decades [1]. It relates a conformal field theory (CFT) to a gravity theory in (asymptotic) Anti-de Sitter (AdS) spacetime. The AdS/CFT correspondence is a particular example of the holographic principle, which states that in a gravitational theory, the number of degrees of freedom in a volume V scale as the surface area ∂V of that volume [2]. The AdS/CFT correspondence is a gauge/gravity duality, where the gravity side is represented by a type IIB superstring theory.

However, we still cannot derive these dualities from first principles. This means that we do not know how spacetime on the gravity side is encoded on the dual CFT side [3]. The emergence of spacetime in the gauge/gravity duality represents one of the main principles in quantum gravity, but the details remain unclear. The radial direction z of AdS probably emerges from the renormalization group scale of the dual field theory, but this duality still is not described in full accuracy [4].

Shortly after the discovery of the AdS/CFT correspondence, there were various attempts to extend it to a dS/CFT correspondence [5, 6, 7]. It is necessary to look for a dS/CFT duality, since our universe is unlikely to have an AdS boundary but may well have a dS boundary in the far future. Recently a concrete example of a dS/CFT correspondence has been conjectured, where the focus was set mainly on the dynamics of such a duality [8]. This conjecture states that the conformal field theory has not to be unitary. The non-unitarity is even rather a condition to relate it to a de Sitter spacetime with a Lorentzian casual structure after Euclideanization [9].

However, it is probable that we have to be very careful with Euclideanizing asymptotic de Sitter spacetime as it differs significantly from Anti-de Sitter spacetime. The main point is that the de Sitter boundary is at temporal infinity and therefore the traditional Euclideanzation procedure is not fully adequate for asymptotic dS spacetimes. Instead of complexifying time, it could be necessary to complexify space. In doing so, we would get a negative Euclidean signature $(- \cdots -)$, whose difference to $(+ \cdots +)$ is not trivial [10, 11].

For the non-unitarian CFT dual to higher spin gravity in dS spacetime it is assumed to use a pseudo-hermitian linear $\mathrm{Sp}(2N)$ model with anticommuting scalars [8, 9, 12, 13]. The recent conjecture of a dS/CFT correspondence [8] is an analogue of the Giombi-Klebanov-Polyakov-Yin (GKPY) duality relating the linear three dimensional $\mathrm{O}(N)$ model to Vasiliev higher spin gravity in AdS_4 [14, 15]. The higher spin gravity may arise as a tensionless limit of the string theory, which enters in the AdS/CFT correspondence on the gravity side. The higher spin (HS) theory was developed by Vasiliev [16] and involves an infinite number of higher spin gauge fields. The use of the HS gravity and the $\mathrm{O}(N)$ model in the AdS/CFT correspondence is very interesting, as both sides are tractable and we can explore first principles of the duality.

In order to consider generalizations of the gauge/gravity duality away from the conformal fixed points and deriving it from field theory, it is possible to interpret it as a geometric realization of the renormalization group (RG) flow, as was done already shortly after the discovery of the AdS/CFT correspondence [17, 18, 19, 20, 21, 22]. Later then it was tried to relate the renormalization group equations to higher spin theory, in a way that the HS equations can be derived from the RG equations [23, 24, 25, 26, 27, 28]. However, there still seem to be some obstacles with the form of the assumed cut-offs in the RG equations [4].

In this thesis we first present the basic ingredients which are needed for the analysis of the mapping of higher spin gravity to AdS and dS spacetime. The basic ingredients on the gravity side are the presentation of the Anti-de Sitter and de Sitter spacetime as well as the introduction to higher spin gravity. On the gauge side, we introduce the linear $\mathrm{O}(N)$ and $\mathrm{Sp}(2N)$ models as well as the renormalization group.

In the third chapter we relate the renormalization group equations of the linear $\mathrm{O}(N)$ model to the higher spin gravity in AdS spacetime. We will start with the RG equation of the $\mathrm{O}(N)$ model, then move to the higher spin equations in order to represent the linearized renormalizations group flow as higher spin equations on AdS_4. In chapter 3.4 we will analyze the gauge transformations and in 3.5 we will add the inhomogenity.

In chapter 4 we will perform the mapping between the conformal algebra and de Sitter spacetime. For this we first analyze the conformal algebra of the $\mathrm{O}(N)$ model. Then we map the conformal algebra to AdS as well as to dS spacetime and find an important difference. Finally in chapter 4.3 we map the conformal algebra of the $\mathrm{Sp}(2N)$ model to de Sitter spacetime.

2 Basic ingredients

In this chapter we introduce the four basic ingredients which are needed to describe the duality between higher spin gravity in AdS and dS spacetime and the renormalization group equations of conformal field theories on the gauge side. On the gravity side of the duality we need maximally symmetric spacetimes with non-zero curvature, therefore, in the first subchapter, we describe the Anti-de Sitter and de Sitter spacetime. In the standard AdS/CFT correspondence the gravity side is represented by a type IIB superstring theory. Here we use a higher spin theory which arises as the tensionless limit of string theory. Therefore we introduce higher spin theory in the second subchapter. On the gauge side we use the linear $\mathrm{O}(N)$ and $\mathrm{Sp}(2N)$ models as the conformal field theories which are linked to the higher spin gravity. These models are presented in subchapter 2.3. Finally we describe the necessary elements of the renormalization group equations which encode the conformal field theories and are linked to the higher spin equations in chapter 3.

2.1 Anti-de Sitter and de Sitter spacetime

The Copernican principle states that our universe is homogenous and isotropic, which implies that space is maximally symmetric. To go beyond this principle would mean to insist that also spacetime is maximally symmetric. These spacetimes could be considered the ground states of general relativity [29]. Symmetries of spacetime are given by Killing vector fields, which are vector fields $X = X^\mu(x)\partial_\mu$ along which the Lie derivative $\mathcal{L}_X$ leaves the metric $g_{\mu\nu}$ invariant, i.e. $\mathcal{L}_X g_{\mu\nu} = 0$. A manifold of dimension d can only have up to $d(d+1)/2$ linearly independent Killing vector fields, i.e. for Minkowski spacetime we have d translational and $d(d-1)/2$ rotational isometries. Spacetimes which satisfy this bound of $d(d+1)/2$ isometries are called maximally symmetric and have a curvature which is the same everywhere [2].

In the case of Riemannian manifolds we have three different maximally symmetric spaces, namely Euclidean, spherical or hyperbolic. Moving to Lorentzian manifolds, the three maximally symmetric spacetimes are the Minkowski, de Sitter (dS) and Anti-de Sitter (AdS) spacetimes. Both the Euclidean and the Lorentzian variants are distinguished by the sign of the curvature, i.e. the Ricci scalar R respectively the cosmological constant Λ_0. If $R = 0$ (equals to $\Lambda_0 = 0$), then we obtain Minkowski spacetime. For non-zero curvature, our maximally symmetric spacetime is either de Sitter for $R > 0$ ($\Lambda_0 > 0$) or Anti-de Sitter spacetime for $R < 0$ ($\Lambda_0 < 0$).

The $(d+1)$-dimensional Anti-de Sitter spacetime (AdS_{d+1}) can be embedded into the $(d+2)$-dimensional Minkowski spacetime $(X^0, X^1, \ldots, X^d, X^{d+1}) \in \mathbb{R}^{d,2}$ with $\eta_{MN} = \mathrm{diag}(-1, 1, \ldots, 1, -1, 1)$ and $M, N \in \{0, \ldots, d+1\}$ [2].[1] This means that

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + \cdots - (dX^d)^2 + (dX^{d+1})^2 \equiv \eta_{MN} dX^M dX^N \ . \tag{2.1}$$

For $\eta_{MN} = \mathrm{diag}(-1, 1, \ldots, 1, -1, 1)$ the AdS_{d+1} is given by the hypersurface

$$-(X^0)^2 + (X^1)^2 + \cdots + (X^{d-1})^2 - (X^d)^2 + (X^{d+1})^2 = -L^2 \tag{2.2}$$

[1] The second minus in η_{MN} was set to the penultimate position in order to obtain an easier parametrization of the dS spacetime in (2.7) and later advantages in chap. 4.3.

with the radius of curvature L.
To get from here to the Poincaré patch[2] with the z-coordinates $z \in \mathbb{R}_0^+, t \in \mathbb{R}$ and $\vec{x} = (x^1, x^2, \ldots, x^{d-1}) \in \mathbb{R}^{d-1}$ we need the following parametrization

$$X^0 = \frac{z}{2} + \frac{1}{2z}(\vec{x}^2 - t^2 + L^2) , \qquad X^i = \frac{Lx^i}{z} ,$$
$$X^d = \frac{Lt}{z} , \qquad X^{d+1} = \frac{z}{2} + \frac{1}{2z}(\vec{x}^2 - t^2 - L^2) , \tag{2.3}$$

with $i = 1, 2, \ldots, d-1$. Inserting the parametrization into (2.1) leads to

$$\begin{aligned} ds^2 &= \left[-\left(\frac{1}{2} - \frac{1}{2z^2}(\vec{x}^2 - t^2 + L^2)\right)^2 + \frac{L^2\vec{x}^2}{z^4} - \frac{L^2t^2}{z^4} + \left(\frac{1}{2} - \frac{1}{2z^2}(\vec{x}^2 - t^2 - L^2)\right)^2 \right] dz^2 \\ &\quad + \left[-\frac{t^2}{z^2} - \frac{L^2}{z^2} + \frac{t^2}{z^2} \right] dt^2 + \left[-\frac{\vec{x}^2}{z^2} + \frac{L^2}{z^2} + \frac{\vec{x}^2}{z^2} \right] d\vec{x}^2 \\ &= \left[-\left(-\frac{2L^2}{4z^2} \right) + \frac{2L^2}{4z^2} \right] dz^2 - \frac{L^2}{z^2} dt^2 + \frac{L^2}{z^2} d\vec{x}^2 \\ &= \frac{L^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2 \right) = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right) , \end{aligned} \tag{2.4}$$

where the conformal boundary is located at $z = 0$ and the Poincaré horizon at $z \to \infty$. We use the mostly plus convention for the metric with $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$ with $\mu, \nu \in \{0, 1, \ldots, d-1\}$.

In the defining equation (2.2) the AdS_{d+1} bears the isometry group $\text{SO}(d, 2)$ and acts on the AdS boundary as the conformal group of Minkowski space. However, in the Poincaré coordinates only the subgroups $\text{ISO}(d-1, 1)$ and $\text{SO}(1, 1)$ are manifest. The Poincaré group $\text{ISO}(d-1, 1)$ includes all Poincaré transformations acting on $(t, \vec{x})$ on the conformal boundary of AdS and can be identified with the Lorentz transformation $L_{\mu\nu}$ and the translation P_μ. The second subgroup is $\text{SO}(1, 1)$ which acts as $(t, \vec{x}, z) \mapsto (\lambda t, \lambda\vec{x}, \lambda z)$ and can be identified with the dilatation operator D of the conformal symmetry group of $\mathbb{R}^{d-1,1}$ [2].

Analoguously, the $(d+1)$-dimensional de Sitter spacetime (dS_{d+1}) may be embedded into the higher dimensional Minkowski spacetime with $\eta_{MN} = \text{diag}(-1, 1, \ldots, 1)$ and line element

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + \cdots + (dX^{d+1})^2 \equiv \eta_{MN} dX^M dX^N . \tag{2.5}$$

The de Sitter spacetime is given by the hypersurface

$$-(X^0)^2 + (X^1)^2 + \cdots + (X^d)^2 + (X^{d+1})^2 = L^2 \tag{2.6}$$

where the sign in front of the squared curvature now is flipped compared to (2.2).

To transform the de Sitter spacetime to the Poincaré patch[3] with the conformal time coor-

[2] $z \in \mathbb{R}_0^+$ covers just one half of the AdS spacetime. We could also select $z \in \mathbb{R}_0^-$. To cover AdS we need both. $z \to \pm\infty$ is identified with the Killing horizon, i.e. a null hypersurface with $k_\mu k^\mu = 0$, with k_μ being a Killing vector.

[3] As seen above for AdS, in dS we also could select $\tau \in \mathbb{R}_0^+$. We need both to cover dS.

dinate $\tau \in \mathbb{R}_0^-$ and $\vec{x} = (x^1, \ldots, x^d) \in \mathbb{R}^d$ we use the following parametrization

$$\begin{aligned} X^0 &= \frac{\tau}{2} - \frac{1}{2\tau}(\vec{x}^2 + L^2) \ , \\ X^i &= \frac{Lx^i}{\tau} \ , \\ X^{d+1} &= \frac{\tau}{2} - \frac{1}{2\tau}(\vec{x}^2 - L^2) \ , \end{aligned} \tag{2.7}$$

with $i = 1, 2, \ldots, d$. Inserting the coordinate transformation (2.7) into (2.5) we obtain

$$\begin{aligned} ds^2 &= \left[-\left(\frac{1}{2} + \frac{1}{2\tau^2}(\vec{x}^2 + L^2)\right)^2 + \frac{L^2\vec{x}^2}{\tau^4} + \left(\frac{1}{2} + \frac{1}{2\tau^2}(\vec{x}^2 - L^2)\right)^2 \right] d\tau^2 + \left[-\frac{\vec{x}^2}{\tau^2} + \frac{L^2}{\tau^2} + \frac{\vec{x}^2}{\tau^2} \right] d\vec{x}^2 \\ &= \left[-\left(\frac{2L^2}{4\tau^2}\right) - \frac{2L^2}{4\tau^2} \right] d\tau^2 + \frac{L^2}{\tau^2} d\vec{x}^2 = \frac{L^2}{\tau^2}\Big(-d\tau^2 + d\vec{x}^2 \Big) \ . \end{aligned} \tag{2.8}$$

Now spatially there is no boundary, and spacetime only ends in (conformal) time at the future boundary $\tau = 0$ and the past horizon $\tau \to -\infty$ [8, 30].

The transformation from AdS_{d+1} to dS_{d+1} in Poincaré patch can be done via a double Wick rotation (compare [8]). We start with AdS_{d+1}

$$ds^2 = \frac{L_{\mathrm{AdS}}^2}{z^2}\Big(dz^2 - dt^2 + d\vec{x}^2\Big)$$

and set

$$z = -i\tau \quad , \quad t = ix^d \quad \text{as well as} \quad L_{\mathrm{AdS}} = -iL_{\mathrm{dS}} \tag{2.9}$$

and obtain dS_{d+1}

$$ds^2 = \frac{L^2}{\tau^2}\Big(-d\tau^2 + d\vec{x}^2 \Big) \ .$$

Thus in transforming from AdS_{d+1} to dS_{d+1} the double Wick rotation (2.9) means that the coordinate z, representing a warped direction is transformed into a conformal time coordinate τ and the standard time coordinate t is transformed into the space coordinate x^d.

dS_{d+1} has the isometry group $\mathrm{SO}(d+1, 1)$ in its defining equation (2.6). In Poincaré coordinates it also has two subgroups, the Euclidean group $\mathrm{ISO}(d)$ which is relatable to Euclidean translations, as well as $\mathrm{SO}(1, 1)$ which acts as $(\tau, \vec{x}) \mapsto (\lambda\tau, \lambda\vec{x})$ and can be identified with a Dilatation operator related to τ instead of t.

In this context it should be noted that the conformal group $\mathrm{C}(d-1, 1)$, which is the invariance group of the light-cone, i.e. its transformations leave $ds^2 = 0$ invariant, is isomorphic to $\mathrm{SO}(d, 2)$ of which $\mathrm{SO}(d-1, 1)$ is a subgroup [31, 32]. The last statement as well as the conformal transformations are only valid for $d > 2$. Furthermore $\mathrm{SO}(4, 1)$ is isomorphic to the symplectic group $\mathrm{Sp}(2, 2)$ [33]. For more details about orthogonal and symplectic groups we refer to appendix B.

2.2 Higher spin gravity

Higher spin gravity was formulated by Vasiliev in 1990 [16]. It is an extension of gravity that also includes a massless scalar as well as massless fields with spins $s = 3, 4, \ldots \infty$, each occurring once. Higher spin fields are given in terms of higher spin representations of the Poincaré group [34].

The interest for consistent interactions of massless higher spin fields grew in the 70s in relation to supergravity. The aims were to better understand supergravity and to improve its quantum behaviour by the inclusion of higher spin gauge fields as well as theoretical viewpoints, to better understand gauge theories going beyond Yang-Mills and including massless fields of arbitrary spin [34].

However, most of these efforts were fruitless, until in 1978 Flato and Fronsdal [35] discovered that the symmetric product of two ultra-short representations (singletons) of the AdS group $\mathrm{SO}(d, 2)$ in four dimensions includes an infinite tower of massless higher spin representations. As a byproduct a free field equation for massless fields of arbitrary spin in AdS_4 was constructed. In 1987, Fradkin and Vasiliev [36] arrived at an interacting higher spin gauge theory.

If the theory is based on an infinite dimensional extension of the AdS_4 algebra and if the interaction is expanded around the AdS_4 background, then the gravitational interaction of massless higher spin fields does exist. For the development of higher spin fields we need a suitable choice of higher spin symmetry algebra. As a result, the flat space limit cannot be taken in the interactions involving higher spin fields, but we need AdS or dS spacetime [34].

Higher spin gravity as a tensionless limit of string theory

Before elaborating on higher spin gravity we will explain its link to string theory. Higher spin theory is related to string theory since it may be viewed as its tensionless limit [3], where the string length l_s is taken to be infinite. The justification for this statement will be sketched in the following. In string theory the fundamental objects are one-dimensional extended strings which sweep out a $(1 + 1)$-dimensional worldsheet Σ and therefore the string theory is non-local [2]. The worldsheet is parametrized by the proper time τ and the spatial extend σ of the string. The simplest action for strings invariant to parametrization is the Nambu-Goto action

$$\mathcal{S}_{NG} = -\frac{1}{2\pi\alpha'} \int\limits_{\Sigma} d\tau d\sigma \sqrt{-\det(\partial_\alpha X^M \partial_\beta X^N \eta_{MN})} \tag{2.10}$$

where α' is the Regge slope. Due to the square root in (2.10) it is very difficult to quantize the theory based on $\mathcal{S}_{NG}$. However, we can introduce a worldsheet metric $h_{\alpha\beta}(\tau, \sigma)$ as an auxiliary field. The dynamics of the string are then given by the Polyakov action

$$\mathcal{S}_P = -\frac{1}{4\pi\alpha'} \int\limits_{\Sigma} d\tau d\sigma \sqrt{-h}\; h^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \eta_{MN} \tag{2.11}$$

where $h = \det(h_{\alpha\beta})$. $\mathcal{S}_{NG}$ and $\mathcal{S}_P$ are equivalent on classical level but the theory specified by the latter is easier to quantize. After quantization the different oscillation modes can be interpreted as particles. The Regge slope α' is the slope of the Regge trajectory where the growing masses square M^2 and the spins J of the spectrum of the strings are plotted against each other [37] with a possible correction term M_0, i.e.

$$M^2 = \frac{1}{\alpha'} J - M_0^2 \ . \tag{2.12}$$

The string length l_s and the string tension T is related to the Regge slope α' via

$$\alpha' = l_s^2 \qquad \text{and} \qquad T = \frac{1}{2\pi\alpha'} \ . \tag{2.13}$$

And as higher spin theory comes up with fields with higher spins which have no mass, i.e. $M = 0$, we can conclude with (2.12) that the Regge slope α' and consequently the string length l_s is going to infinity. This would mean that higher spin gravity is very non-local and that even the casual structure of spacetime is changed and we need a new notion of geometry [3, 34].

Formulation of higher spin gravity

Here, we present the formulation of the higher spin theory in asymptotic AdS_4 spacetime pursuant to Vasiliev and collaborators [16, 36]. We will denote the spacetime coordinates by x^m where $m \in \{0, 1, 2, 3\}$.

The degrees of freedom of higher spin gravity can be recast into master fields which contain an infinite tower of higher spin degrees of freedom. In the Vasiliev theory the master fields are a spacetime one-form W_m, i.e. $W = W_m \, dx^m$ as well as a spacetime zero-form B. All these master fields depend on the spacetime coordinates x^m themselves as well as on a set of additional variables which we will discuss below. First, we discuss the meaning[4] of W_m and B. While W_m encodes the gravitational spin 2 as well as the spin $s = 3, 4, \ldots$ degrees of freedom, the master field B represents the scalar field as well as its derivatives.

The tower of higher spin degrees of freedom are encoded in the master field by using variables $Y = (y_\alpha, \bar{y}_{\dot{\alpha}})$ with $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$. We can expand the master fields into a series of y_α and $\bar{y}_{\dot{\alpha}}$, i.e. for W we obtain

$$\begin{aligned} W(x, y_\alpha, \bar{y}_{\dot{\alpha}}) = {} & W^{\alpha\beta}(x) y_\alpha y_\beta + W^{\alpha\dot{\beta}}(x) y_\alpha \bar{y}_{\dot{\beta}} + W^{\dot{\alpha}\dot{\beta}}(x) \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} \\ & + W^{\alpha\beta\gamma\delta}(x) y_\alpha y_\beta y_\gamma y_\delta + \text{all combinations of 4 undotted/dotted indices} \\ & + \text{terms with 6,8,10,... indices} \ . \end{aligned} \tag{2.14}$$

The coefficients in front are functions which may depend only on spacetime coordinates and hence correspond to higher spin fields. The following picture emerges: The component of W with $(2s - 2)$ indices α and $\dot{\alpha}$ corresponds to a spin s field in spacetime.

The variables $(y_\alpha, \bar{y}_{\dot{\alpha}})$ are referred to as twistors. In other words, the variables are spinors which do not commutate and satisfy the following commutation relations[5]

$$[y_\alpha, y_\beta]_* = 2i\epsilon_{\alpha\beta} \quad , \quad [\bar{y}_{\dot{\alpha}}, \bar{y}_{\dot{\beta}}]_* = 2i\epsilon_{\dot{\alpha}\dot{\beta}} \quad , \quad [y_\alpha, \bar{y}_{\dot{\beta}}]_* = 0 \, . \tag{2.15}$$

[4] Note that the equations of motion, which we will present later, are highly non-linear. Hence, in order to disentangle and interprete the degrees of freedom we first have to linearise the higher spin theory around AdS_4.

[5] Technically speaking since they satisfy commutation relations (as opposed to anticommutation relations) they are Grassmann even.

Also note that there exist two different multiplication rules for y_α: we can just multiply them, i.e. $y_\alpha\, y_\beta$, or we can use the star product, i.e. $y_\alpha * y_\beta$, which was already used in the commutators (2.15) and which is defined by

$$(f * g)(y) = \frac{1}{(2\pi)^2} \int d^2u\; d^2v\; e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}} f(y+u)g(y+v) = f(y) e^{i\epsilon_{\alpha\beta} \frac{\overleftarrow{\partial}}{\partial y_\alpha} \frac{\overrightarrow{\partial}}{\partial y_\beta}} g(y)\,. \tag{2.16}$$

Hence we obtain for $y_\alpha * y_\beta$ the following expression

$$y_\alpha * y_\beta = \frac{1}{(2\pi)^2} \int d^2u d^2v e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}} (y_\alpha + u_\alpha)(y_\beta + v_\beta) = y_\alpha \exp\left(i \frac{\overleftarrow{\partial}}{\partial y_\gamma} \epsilon_{\gamma\delta} \frac{\overrightarrow{\partial}}{\partial y_\delta} \right) y_\beta \tag{2.17}$$

and therefore

$$y_\alpha * y_\beta = y_\alpha y_\beta + i\epsilon_{\alpha\beta}\ .$$

We can also define an analogous star product multiplication rule for $\bar{y}_{\dot\alpha}$. In particular we obtain

$$\bar{y}_{\dot\alpha} * \bar{y}_{\dot\beta} = \bar{y}_{\dot\alpha}\bar{y}_{\dot\beta} + i\epsilon_{\dot\alpha\dot\beta}\,, \qquad y_\alpha * \bar{y}_{\dot\beta} = y_\alpha \bar{y}_{\dot\beta}\ .$$

The Weyl (Moyal) star-product allows us to rewrite theories depending on non-commuting coordinates in terms of commuting ones [38].

Note that y_α generates the Lie algebra $\mathfrak{sp}(2)$ using the star-product as we can see from (2.15). The explicit calculation of (2.15) and (2.16) is shown in appendix C.1. We can repeat the same arguments for $y_{\dot\alpha}$. In total, $Y = (y_\alpha, \bar{y}_{\dot\alpha})$ forms a representation of $\mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$ which is isomorphic to the generators of the isometry group of AdS_4, i.e. to $\mathfrak{so}(3,2)$.[6] Hence we see that the additional variables $Y = (y_\alpha, \bar{y}_{\dot\alpha})$ play an important role: first they can be used to decompose the master field into its spin components and furthermore they imprint the (asymptotic) AdS_4 geometry into higher spin gravity.

Besides the additional variables $Y = (y_\alpha, \bar{y}_{\dot\alpha})$ we also have to include another set of twistor variables $Z = (z_\alpha, \bar{z}_{\dot\alpha})$ with commutation relations

$$[z_\alpha, z_\beta]_* = -2i\epsilon_{\alpha\beta} \quad , \quad [\bar{z}_{\dot\alpha}, \bar{z}_{\dot\beta}]_* = -2i\epsilon_{\dot\alpha\dot\beta} \quad , \quad [z_\alpha, \bar{z}_{\dot\beta}]_* = 0\ . \tag{2.18}$$

These additional twistor variables are useful when writing the equations of motion in terms of the master fields. However, they do not incorporate any physical degrees of freedom and hence should be viewed as auxiliary variables. In other words: the two master fields $W(x|Y,Z)$ and $B(x|Y,Z)$ depend on the spacetime coordinates x^m as well as the twistor variables $Y = (y_\alpha, \bar{y}_{\dot\alpha})$ and $Z = (z_\alpha, \bar{z}_{\dot\alpha})$. As explained above, Y plays an crucial role. This is not true for Z: all physical degrees of freedom will be contained in the master fields if we restrict them to $z_\alpha = z_{\dot\alpha} = 0$.

The dynamics of field theories are usually given in terms of an action principle. However, for higher spin gravity, there is no consistent action principle known [34,39]. Hence, we only display the equations of motion [40]

$$dB + W * B - B * W = 0, \qquad dW + W \wedge * W = 0\ . \tag{2.19}$$

[6] As these are complex algebras, $\mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$ is also isomorphic to generators of the isometry group of dS_4, i.e. to $\mathfrak{so}(4,1)$. Compare chapter 2.1.

The first equation of (2.19) states that the exterior covariant derivative D of the field B should vanish, also called zero-torsion condition, while the second equation is the flatness condition [40].

The Vasiliev higher spin gravity includes three more equations, as well as an auxiliary field $S = s_\alpha\, dz^\alpha + s_{\dot\alpha}\, d\bar z^{\dot\alpha}$. Note that S is a spacetime zero-form, but a one-form in the auxiliary variables $Z = (z_\alpha, \bar z_{\dot\alpha})$. Hence, this field will not contain physical degrees of freedom. The remaining equations of motion involving S are given by

$$\begin{aligned} S * S &= -idz_\alpha dz^\alpha(1 + F(B) * \kappa) - id\bar z_{\dot\alpha} d\bar z^{\dot\alpha}(1 + \bar F(B) * \bar\kappa)\,, \\ S * B - B * S &= 0\,, \\ dS + W * S - S * W &= 0\ . \end{aligned} \tag{2.20}$$

In (2.20) $F(B)$ is a particular function of B with respect to the star product and $\kappa = k\,K$ where k is a Kleinian with

$$kf(z^\alpha, dz^\alpha, y^\alpha, \bar z^{\dot\alpha}, d\bar z^{\dot\alpha}, \bar y^{\dot\alpha}) = f(-z^\alpha, -dz^\alpha, -y^\alpha, \bar z^{\dot\alpha}, d\bar z^{\dot\alpha}, \bar y^{\dot\alpha})k \tag{2.21}$$

and $K = e^{iz_\alpha y^\alpha}$ an inner Kleinian for the star-product with

$$f(y, z) * K = f(-z, -y)K \quad , \quad K * f(y, z) = Kf(z, y)\ . \tag{2.22}$$

Linearizing the equations of motion to first order in the scalar field B, the effect of S is that W and B are independent of the Kleinian k and the auxiliary spinor variables Z.

The five equations of motion (2.19) and (2.20) are invariant under higher spin gauge transformations. According to [40] higher spin gauge transformations read

$$\delta B(x|Y, Z) = [B(x|Y, Z), \epsilon(x|Y, Z)]_* \equiv [B, \epsilon]_* \tag{2.23}$$

$$\delta W(x|Y, Z) = d\epsilon(x|Y, Z) + [W(x|Y, Z), \epsilon(x|Y, Z)]_* \equiv d\epsilon + [W, \epsilon]_*. \tag{2.24}$$

where ϵ is the infinitesimal gauge parameter, which is itself a master field. It does not only contain diffeomorphisms being spin 2 gauge transformations but also their higher spin counterparts.

To get familiar with the higher spin equations we explicitly check if these are invariant under (2.23) and (2.24), having in mind that B and ϵ are 0-forms, whereas $W = W_m dx^m$ and $d\epsilon$ are 1-forms. For the transformations we define:

$$\tilde B \equiv B + \delta B = B - \epsilon * B + B * \epsilon \quad , \quad \tilde W \equiv W + \delta W = W + d\epsilon + W * \epsilon - \epsilon * W\ . \tag{2.25}$$

We now replace B and W with $\tilde B$ and $\tilde W$ in (2.19) and receive

$$\begin{aligned} d\tilde B + \tilde W * \tilde B - \tilde B * \tilde W &= dB - d(\epsilon * B) + d(B * \epsilon) \\ &\quad + (W + d\epsilon + W * \epsilon - \epsilon * W) * (B - \epsilon * B + B * \epsilon) \\ &\quad - (B - \epsilon * B + B * \epsilon) * (W + d\epsilon + W * \epsilon - \epsilon * W) \\ &= dB + W * B - B * W - d\epsilon * B - \epsilon * dB + dB * \epsilon + B * d\epsilon \\ &\quad - W * \epsilon * B + W * B * \epsilon + d\epsilon * B + W * \epsilon * B - \epsilon * W * B \\ &\quad - B * d\epsilon - B * W * \epsilon + B * \epsilon * W + \epsilon * B * W - B * \epsilon * W \\ &= -\epsilon * (dB + W * B - B * W) + (dB + W * B - B * W) * \epsilon \\ &= dB + W * B - B * W \end{aligned} \tag{2.26}$$

and by using $d(\epsilon \wedge *W) = dW \wedge *\epsilon - W \wedge *d\epsilon$

$$\begin{aligned} d\tilde{W} + \tilde{W} \wedge *\tilde{W} &= dW + d(d\epsilon) + d(W \wedge *\epsilon) - d(\epsilon \wedge *W) \\ &\quad + (W + d\epsilon + W \wedge *\epsilon - \epsilon \wedge *W) \wedge *(W + d\epsilon + W \wedge *\epsilon - \epsilon \wedge *W) \\ &= dW + W \wedge *W + d(d\epsilon) + dW \wedge *\epsilon - W \wedge *d\epsilon - d\epsilon \wedge *W - \epsilon \wedge *dW \\ &\quad + W * d\epsilon + W \wedge *W * \epsilon - W * \epsilon * W + d\epsilon * W + W * \epsilon * W - \epsilon * W \wedge *W \\ &= (dW + W \wedge *W) * \epsilon - \epsilon * (dW + W \wedge *W) = dW + W \wedge *W \end{aligned} \tag{2.27}$$

and see that they are invariant.

AdS$_4$ vacuum

Above we introduced the Vasiliev higher spin equations in a background independent manner. In this form we cannot yet see that the higher spin equations describe higher spin gauge fields in AdS$_4$. To be able to formulate perturbation theory, we have to expand the fields around some background that solves the equations of motion [41]. The maximally symmetric vacuum solution describing AdS$_4$ spacetime reads

$$W_0(x|Y) = e_0(x|Y) + \omega_0(x|Y) = (e_0)_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} + (\omega_0)_{\alpha\beta} y^\alpha y^\beta + (\omega_0)_{\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}} , \tag{2.28}$$

where we set $B = 0$. Therefore we have a spin 2 metric as background. In (2.28) e_0 and ω_0 are the vielbein and the spin connection 1-forms on AdS$_4$. As $ds^2 \equiv g_{mn} dx^m dx^n$ and $g_{mn} = e^a_m e^b_n \eta_{ab}$, we can write the AdS$_4$ metric as [2]

$$ds^2 = \frac{L^2}{z^2}\Big(dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu\Big) = e^a_m e^b_n dx^m dx^n \eta_{ab} , \tag{2.29}$$

where $\eta_{ab} = \mathrm{diag}(1, -1, 1, 1)$. The components of the vielbein are $e^z = \frac{L}{z} dz$, $e^t = \frac{L}{z} dt$, $e^{x_1} = \frac{L}{z} dx_1$ and $e^{x_2} = \frac{L}{z} dx_2$ with $ds^2 = e^z e^z - e^t e^t + e^{x_1} e^{x_1} + e^{x_2} e^{x_2}$. The spin connection components can be derived via $de^a + \omega^a{}_b \wedge e^b = 0$ [2].

In Poincaré coordinates we can write e_0 and ω_0 explicitly with $L = 1$ as [26]

$$e_0(x|Y) = -\frac{1}{4i}\frac{dx^m}{z}(\sigma_m)_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} , \quad \omega_0(x|Y) = -\frac{1}{4i}\frac{dx^\mu}{z}((\sigma^{\mu z})_{\alpha\beta} y^\alpha y^\beta + (\bar{\sigma}^{\mu z})_{\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}}) \tag{2.30}$$

with the 4D sigma matrices $(\sigma_m)_{\alpha\dot{\beta}} = (1, \vec{\sigma})_{\alpha\dot{\beta}}$, $(\bar{\sigma}_m)^{\dot{\alpha}\beta} = (1, -\vec{\sigma})^{\dot{\alpha}\beta}$. m, n are the $d + 1 = 4$ dimensional bulk indices with $m, n \in \{0, 1, 2, 3\}$. For conventions regarding the sigma matrices we refer to appendix A. Comparing (2.28) and (2.30) with the first line of the series expansion (2.14), we see that W_0 includes only two lower indices and thus represents the spin 2 fields, hence a background metric.

2.3 Linear O(*N*) and Sp(2*N*) models

Linear O(*N*) model

A theory which is invariant under O(N) transformations[7] (as well as Poincaré transformations) is the N-component scalar field theory which has the action

$$\mathcal{S}_\phi = \int d^dx \mathcal{L}_\phi = \int d^dx \Big(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^i(x)\partial_\nu\phi^i(x) - \frac{1}{2}m^2\phi^i(x)^2 \Big) \ , \tag{2.31}$$

with $i = 1, \ldots, N$ in d-dimensional Minkowski spacetime using $d^dx \equiv d^{d-1}\vec{x}\ dt$ and $(x) \equiv (\vec{x}, t)$. It is called the linear O(N) model. From now on we suppress the component indices i.

The Noether theorem states that every continuous symmetry is associated with a conserved current. For spacetime translations a^μ the conserved current is the canonical energy-momentum tensor

$$T^\mu{}_\nu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi + \mathcal{L}\delta^\mu_\nu \ . \tag{2.32}$$

Here we already use the canonical energy-momentum tensor $T^\mu{}_\nu$ instead of the energy-momentum tensor $\Theta^\mu{}_\nu$ which is not necessarily symmetric by construction. $T^\mu{}_\nu$ is still conserved but is also symmetric. Both are related through $T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}$ with $f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$. For the O($N$) model the energy momentum tensor is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + \eta_{\mu\nu}\mathcal{L}_\phi \ . \tag{2.33}$$

The associated Noether charge of the energy-momentum tensor is

$$P_\mu = \int d^{d-1}\vec{x}\ T^0{}_\mu \ . \tag{2.34}$$

For time translations we get the Hamiltonian

$$H \equiv -P_0 = -\int d^{d-1}\vec{x}\ T^0{}_0 = \int d^{d-1}\vec{x}\ T_{00} \ , \tag{2.35}$$

and for space translations we get the physical momentum

$$P_i = \int d^{d-1}\vec{x}\ T^0{}_i \ . \tag{2.36}$$

For the O(N) model with action (2.31) we obtain for the Hamiltonian H

$$H = \int d^{d-1}\vec{x}\bigg(\frac{1}{2}\pi(x)^2 + \frac{1}{2}(\vec{\nabla}\phi(x))^2 + \frac{1}{2}m^2\phi(x)^2\bigg) \tag{2.37}$$

with $\partial_0\phi(x) = \pi(x)$ and for the physical momentum P_i

$$P_i = -\int d^{d-1}\vec{x}\ \pi(x)\partial_i\phi(x) \ . \tag{2.38}$$

The canonical commutation relations for quantizing the O(N) model are

$$\Big[\phi(\vec{x}, t), \pi(\vec{y}, t)\Big] = i\delta^{d-1}(\vec{x} - \vec{y}) \tag{2.39}$$

[7] For basics about orthogonal groups O(N) see appendix B.

where we still suppressed the component indices i.

We will now introduce the conformal transformations and their generators in $d > 2$ dimensions, as they will play an important role in the upcoming sections. These are the translation a^μ, the Lorentz transformation $\Lambda^\mu{}_\nu x^\nu$ (with $\Lambda_{\mu\nu} = -\Lambda_{\nu\mu}$), the dilatation λx^μ and the special conformal transformation $b^\mu x^2 - 2(b\cdot x)x^\mu$ corresponding to the operators P_μ, $L_{\mu\nu}$, D and K_μ. If a field theory is invariant under these conformal transformations, it is named a conformal field theory (CFT). A necessary condition of a conformal field theory is its scale invariance, i.e. the vanishing of the β-function in the renormalization group (see below in chapter 2.4). It is possible for a QFT to be scale invariant and not conformally-invariant at the same time, but examples are rare. Scale invariance is part of the larger symmetry, the conformal symmetry group SO($d, 2$).

As we have seen, for translations a^μ the conserved current is the energy-momentum Tensor $T^\mu{}_\nu$ in (2.32) and its Noether charge is P_μ. For the three other conformal transformations we also can calculate conserved currents. For the Lorentz transformation $\Lambda^\mu{}_\nu x^\nu$ the conserved current is $N_{\mu\nu\rho} = x_\nu T_{\mu\rho} - x_\rho T_{\mu\nu}$. However, $N_{\mu\nu\rho}$ is just conserved if $T_{\mu\nu}$ is symmetric, i.e. $T_{\mu\nu} = T_{\nu\mu}$. In massless theories for the dilatation λx^μ the conserved current is $J_{(D)\mu} = x^\nu T_{\mu\nu}$ and for the special conformal transformation $b^\mu x^2 - 2(b\cdot x)x^\mu$ it is $J_{(K)\nu\mu} = x^2 T_{\nu\mu} - 2x_\mu x^\sigma T_{\nu\sigma}$. Here again, we have a restriction. If the theory should be invariant under scale transformations, i.e. if $J_{(D)\mu}$ is conserved, the energy-momentum tensor has to be traceless. A traceless energy-momentum tensor is also needed for $J_{(K)\nu\mu}$ to be conserved.

As the standard energy-momentum tensor is not necessarily traceless, we have to add correction terms to the currents $J_{(D)\mu}$ and $J_{(K)\mu\nu}$ in a way that they get conserved. We will use this method in chapter 4.1 for verifying the conformal algebra of the O(N) model. Another option is to use the improved energy-momentum tensor, which by construction is traceless and for the massless free scalar field reads [2]

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\phi^2 \ . \tag{2.40}$$

The above mentioned associated Noether charges correspond to the following generators

$$\begin{aligned} P_\mu &= \int d^{d-1}\vec{x}\; T^0{}_\mu \qquad L_{\nu\rho} = \int d^{d-1}\vec{x}\; (x_\nu T^0{}_\rho - x_\rho T^0{}_\nu) \\ D &= \int d^{d-1}\vec{x}\; x^\rho T^0{}_\rho \qquad K_\mu = \int d^{d-1}\vec{x}\; (x^2 T^0{}_\mu - 2x_\mu x^\sigma T^0{}_\sigma) \ . \end{aligned} \tag{2.41}$$

The canonical dimension (or scaling dimension) δ (or Δ) of a field ϕ describes the transformation of the field under a rescaling of coordinates, i.e. dilatations $x \mapsto x' = \lambda x$. The field transforms as $\phi(x) \mapsto \phi'(x) = \lambda^{-\delta}\phi(x)$. In natural units the action $\mathcal{S}$ has zero mass dimension and length has inverse mass dimension. Therefore due to dimensional analysis the fields ϕ have the canonical dimension $\delta = [\phi(x)]_c = \frac{d-2}{2}$. Taking into account the Fourier transformations the canonical dimension of $\phi(p)$ in the momentum space reads $[\phi(p)] = -(d-\delta) = -\frac{d-2}{2}$.

The conformal algebra, which will be calculated and analyzed in chapter 4, consists of the following commutators

$$\begin{aligned}
&[D,P_\mu]=iP_\mu && [L_{\mu\nu},L_{\rho\sigma}]=-i\Big(\eta_{\mu\sigma}L_{\nu\rho}+\eta_{\nu\rho}L_{\mu\sigma}-\eta_{\nu\sigma}L_{\mu\rho}-\eta_{\mu\rho}L_{\nu\sigma}\Big)\\
&[D,K_\mu]=-iK_\mu && [L_{\nu\rho},K_\mu]=-i\Big(\eta_{\rho\mu}K_\nu-\eta_{\nu\mu}K_\rho\Big)\\
&[K_\mu,P_\nu]=-2i\eta_{\mu\nu}D-2iL_{\mu\nu} && [L_{\nu\rho},P_\mu]=-i\Big(\eta_{\rho\mu}P_\nu-\eta_{\nu\mu}P_\rho\Big)\\
&[D,L_{\mu\nu}]=0 \qquad [P_\mu,P_\rho]=0 && [K_\mu,K_\rho]=0\ .
\end{aligned} \tag{2.42}$$

Linear Sp(2*N*) model

To arrive at the Sp($2N$) model[8] we start with a N-component free complex scalar field χ in d-dimensional Minkowski spacetime with the action

$$\mathcal{S}_\chi=\int d^dx\mathcal{L}_\chi=\int d^dx\Big(-\partial^\mu\bar\chi^i(x)\partial_\mu\chi^i(x)-m^2\bar\chi^i(x)\chi^i(x)\Big) \tag{2.43}$$

with $i=1,\dots,N$. If χ is a Lorentz scalar, then the model is Lorentz invariant and the action has an explicit internal U(N) symmetry irrespective of whether χ is bosonic or fermionic. Whereas if χ is a fermionic (i.e. Grassmann valued) field, then there is furthermore a hidden Sp($2N$) symmetry [13]. To show this symmetry we express each component χ^i and $\bar\chi^i$ by η_1^i and $\bar\eta_2^i$ as

$$\chi^i=\frac{1}{\sqrt2}(\eta_1^i+i\eta_2^i)\quad,\quad \bar\chi^i=\frac{-i}{\sqrt2}(\eta_1^i-i\eta_2^i)\ . \tag{2.44}$$

Note that η_1 and η_2 are also Grassmann valued fields. If we arrange the real fields η_1^i and η_2^i into a $2N$ vector $\eta=(\eta_1^1,\eta_1^2,\dots,\eta_1^N,\eta_2^1,\eta_2^2,\dots,\eta_2^N)^T$ we obtain

$$\sum_i^N\mathrm{Re}(\bar\chi^i\chi^i)=\frac12\eta^T\epsilon_N\eta\ . \tag{2.45}$$

We have in mind that matrices of Sp($2N$) are those elements A which obey

$$A^T\epsilon_NA=\epsilon_N \qquad\text{with}\qquad \epsilon_N=\begin{pmatrix}0&I_N\\-I_N&0\end{pmatrix}\ . \tag{2.46}$$

To see the effect of the Grassmann valued fields χ we demonstrate the calculation of (2.45). We get

$$\sum_i^N\mathrm{Re}(\bar\chi^i\chi^i)=\sum_i^N\mathrm{Re}\Big(\frac{-i}{\sqrt2}(\eta_1^i-i\eta_2^i)\frac{1}{\sqrt2}(\eta_1^i+i\eta_2^i)\Big)=\frac12\sum_i^N(-\eta_2^i\eta_1^i+\eta_1^i\eta_2^i)$$

and

$$\begin{aligned}
\frac12\eta^T\epsilon_N\eta&=\frac12(\eta_1^1,\eta_1^2,\dots,\eta_1^N,\eta_2^1,\eta_2^2,\dots,\eta_2^N)\begin{pmatrix}0&I_N\\-I_N&0\end{pmatrix}(\eta_1^1,\eta_1^2,\dots,\eta_1^N,\eta_2^1,\eta_2^2,\dots,\eta_2^N)^T\\
&=\frac12(\eta_1^1,\eta_1^2,\dots,\eta_1^N,\eta_2^1,\eta_2^2,\dots,\eta_2^N)(\eta_2^1,\eta_2^2,\dots,\eta_2^N,-\eta_1^1,-\eta_1^2,\dots,-\eta_1^N)^T\\
&=\frac12(\eta_1^1\eta_2^1+\eta_1^2\eta_2^2+\dots+\eta_1^N\eta_2^N-\eta_2^1\eta_1^1-\eta_2^2\eta_1^2-\dots-\eta_2^N\eta_1^N)=\frac12\sum_i^N(-\eta_2^i\eta_1^i+\eta_1^i\eta_2^i)\ ,
\end{aligned}$$

[8] For basics about symplectic groups Sp($2N$) see appendix B.

thus both sides match each other. One can see that if η would not be fermionic (Grassmann valued), then both sides of (2.45) would vanish.

The bilinear form η has the symmetry $\eta \mapsto A\eta$ where $A^T \epsilon_N A = \epsilon_N$ is valid, which is the defining relation (2.46) for A being an element of Sp($2N$). Therefore the action $\mathcal{S}_\chi$ with fermionic fields has an Sp($2N$) symmetry and is called the linear Sp($2N$) model [13].

We have seen that the canonical energy-momentum tensor (2.32) is not necessarily symmetric by construction and needs some correction terms $f^{\lambda\mu\nu}$. For the Sp($2N$) model we prefer to use an alternative definition of the energy-momentum tensor via the metric

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\left(\sqrt{-g}\mathcal{L}_{\text{matter}}\right)}{\delta g^{\mu\nu}} \; . \tag{2.47}$$

The advantages of this definition are that it is symmetric in μ and ν by construction and does not need additional terms, it can be generalized to arbitrary curved spacetimes and for gauge fields it is always gauge invariant [2]. From (2.47) we obtain

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_{\text{matter}}\right)}{\delta g^{\mu\nu}} = -2\frac{\delta \mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}} - \frac{2}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \mathcal{L}_{\text{matter}} \tag{2.48}$$

and with [29]

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \tag{2.49}$$

we finally get

$$T_{\mu\nu} = -2\frac{\delta \mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\text{matter}} \tag{2.50}$$

for the energy-momentum tensor.

For the calculation of the energy-momentum tensor of the Sp($2N$) model we take the Lagrangian with suppressed field component indices

$$\mathcal{L}_{\text{matter}} = \mathcal{L}_\chi = -\partial^\mu \bar{\chi} \partial_\mu \chi - m^2 \bar{\chi}\chi = -\frac{1}{2} g^{\mu\nu} \left(\partial_\mu \bar{\chi} \partial_\nu \chi + \partial_\nu \bar{\chi} \partial_\mu \chi\right) - m^2 \bar{\chi}\chi \; . \tag{2.51}$$

Note that the fields χ have the same canonical dimension $\delta = [\chi(x)]_c = \frac{d-2}{2}$ as the fields ϕ.

With (2.50) we obtain

$$T_{\mu\nu} = \partial_\mu \bar{\chi} \partial_\nu \chi + \partial_\nu \bar{\chi} \partial_\mu \chi + g_{\mu\nu} \mathcal{L}_\chi \tag{2.52}$$

and consequently the Hamiltonian reads

$$\begin{aligned} H &= \int d^{d-1}\vec{x}\; T_{00} = \partial_0\bar{\chi}\partial_0\chi - \partial_0\bar{\chi}\partial_0\chi - \left(\partial_0\bar{\chi}\partial_0\chi - \partial_i\bar{\chi}\partial_i\chi - m^2\bar{\chi}\chi\right) \\ &= \int d^{d-1}\vec{x}\Big(\partial_0\bar{\chi}\partial_0\chi + \partial_i\bar{\chi}\partial_i\chi + m^2\bar{\chi}\chi\Big) \; . \end{aligned} \tag{2.53}$$

With the conjugate-momentum fields $\pi = -\partial_0\bar{\chi}$ and $\bar{\pi} = \partial_0\chi$ we may rewrite the Hamiltonian as

$$H = \int d^{d-1}\vec{x}\Big(-\pi(x)\bar{\pi}(x) + \vec{\nabla}\bar{\chi}(x)\vec{\nabla}\chi(x)\Big) = \int d^{d-1}\vec{x}\Big(\bar{\pi}(x)\pi(x) + \vec{\nabla}\bar{\chi}(x)\vec{\nabla}\chi(x)\Big) \; . \tag{2.54}$$

Now we quantize the model with χ being fermionic (Grassmann valued). The fermionic fields χ, $\bar{\chi}$ and its conjugate-momentum fields π, $\bar{\pi}$ obey the canonical anticommutation relations

$$\Big\{\chi(\vec{x},t),\pi(\vec{y},t)\Big\} = \Big\{\bar{\chi}(\vec{x},t),\bar{\pi}(\vec{y},t)\Big\} = i\delta^{d-1}(\vec{x}-\vec{y}) \tag{2.55}$$

where we again suppressed the component indices i. We have in mind that for Grassmann valued fields $\{A,B\} = \{B,A\}$ is valid. The fields have the following mode expansions [13]

$$\chi(x) = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}2\omega_{\vec{k}}}\Big(b^\dagger(\vec{k})e^{ik\cdot x} + a(\vec{k})e^{-ik\cdot x}\Big)\Big|_{k^0=\omega_{\vec{k}}} \tag{2.56}$$

$$\bar{\chi}(x) = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}2\omega_{\vec{k}}}\Big(-b(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\Big)\Big|_{k^0=\omega_{\vec{k}}} \tag{2.57}$$

which are consistent with the equations of motions and where $\omega_{\vec{k}} = \sqrt{\vec{k}^2+m^2}$ and $k\cdot x = -k^0 t + \vec{k}\cdot\vec{x}$. $a^\dagger(\vec{k})$, $b^\dagger(\vec{k})$ and $a(\vec{k})$, $b(\vec{k})$ are the creation and annihilation operators of (anti-)particles with momentum $\vec{k}$. The extra minus sign in the $b(\vec{k})$ term of $\bar{\chi}$ in (2.57) is chosen in order that the anticommutation relations (2.55) lead to the standard non-vanishing canonical relations

$$\Big\{a(\vec{k}),a^\dagger(\vec{k}')\Big\} = \Big\{b(\vec{k}),b^\dagger(\vec{k}')\Big\} = 2\omega_{\vec{k}}2\pi^{d-1}\delta^{d-1}(\vec{k}-\vec{k}')\ . \tag{2.58}$$

The extra minus sign would not be necessary if χ was bosonic. Usually the Hamiltonian is constructed using χ and its hermitian adjoint $\chi^\dagger$. So χ and $\chi^\dagger$ must be combined in such a way that H is hermitian. Note that this is not the case here, due to the extra minus sign in the $b(\vec{k})$ term in the mode expansion of $\bar{\chi}$ in (2.57). In other words, for the fermionic case, unlike for the bosonic case, $\bar{\chi} \neq \chi^\dagger$ is not the hermitian adjoint of χ. In order to relate them, we now introduce a unitary operator C with $C^\dagger C = 1$ and $C = C^\dagger$, which is defined by the properties $CaC = a$, $CbC = -b$, $Ca^\dagger C = a^\dagger$ and $Cb^\dagger C = -b^\dagger$. With this, the relation between $\bar{\chi}$ and χ can be expressed as

$$\bar{\chi} = C\chi^\dagger C \tag{2.59}$$

which is shown in the following:

$$\bar{\chi}(x) = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}2\omega_{\vec{k}}}\Big(-b(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\Big)\Big|_{k^0=\omega_{\vec{k}}} \tag{2.60}$$

$$= \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}2\omega_{\vec{k}}}\Big(Cb(\vec{k})Ce^{-ik\cdot x} + Ca^\dagger(\vec{k})Ce^{ik\cdot x}\Big)\Big|_{k^0=\omega_{\vec{k}}} \tag{2.61}$$

$$= \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}2\omega_{\vec{k}}}C\Big(b^\dagger(\vec{k})e^{ik\cdot x} + a(\vec{k})e^{-ik\cdot x}\Big)^\dagger C\Big|_{k^0=\omega_{\vec{k}}} = C\chi^\dagger(x)C\ . \tag{2.62}$$

We now show that the Hamiltonian of the Sp(2N) is pseudo-hermitian as it satisfies the intertwined hermiticity condition $H^\dagger = CHC$. These kind of Hamiltonians were already considered by Pauli [42] and more recently by Mostafazadeh [43] in connection with $\mathcal{PT}$ symmetric quantum mechanics. Due to (2.59) and the properties of C follows $(\bar{\chi}\chi)^\dagger = \chi^\dagger\bar{\chi}^\dagger = C\bar{\chi}CC\chi C = C\bar{\chi}\chi C$.

From this we conclude directly that the Hamiltonian (2.53) of the Sp($2N$) model with fermionic fields satisfies the intertwined hermiticity condition

$$H^\dagger = CHC \tag{2.63}$$

and is called pseudo-hermitian. A pseudo-hermitian Hamiltonian can still define unitary quantum mechanics if the inner product is defined as $\langle\psi'|\psi\rangle_c \equiv \langle\psi'|C|\psi\rangle$ [13].

In chapter 3 we will see that the renormalization group flow of the O(N) model can be related to the Anti-de Sitter spacetime in higher spin gravity. Later in chapter 4 we will present a way to relate the Sp($2N$) model to the de Sitter spacetime.

2.4 Renormalization group

The role of the renormalization group (RG) corresponds to varying the cut-off scale Λ which can also be expressed through the renormalization time $t = -\ln(\Lambda/\Lambda_0)$. As quantum field theories need a cut-off in their formulation, they have to be considered as belonging to an infinite dimensional space which may be parametrized in terms of couplings associated with all scalar operators consistent with the basic symmetries of the theory. The renormalization group equations describe the flows in this space of quantum field theories under changes in the cut-off scale Λ. The flow is determined by the requirement that physical observables are independent of the cut-off, at least in the neighbourhood of fixed points, for energies much lower than the cut-off [44]. A fixed point is a point where the RG flow equation, i.e. the β-function vanishes. The β-function reads

$$\frac{\partial g(u)}{\partial \ln(u)} = u\frac{\partial g(u)}{\partial u} = \beta(g(u)) \tag{2.64}$$

where g is the coupling and u the scale.

The gauge/gravity correspondence relates a d-dimensional conformal field theory (CFT) to the geometry of an (Anti)-de Sitter space in $d+1$-dimensions. The extra dimension is related to the energy scale (cut-off scale) of the CFT. The holographic duality is a geometrization of quantum dynamics of systems with a large number of degrees of freedom. Let us have a look on the duality with the Kadanoff-Wilson renormalization group approach to the analysis of lattice systems [37, 45, 46]. The minimal lattice spacing corresponds to the cut-off scale Λ. We consider a non-gravitational system in a lattice with lattice spacing a and a Hamiltonian

$$H = \sum_{x,i} g_i(x,a)\mathcal{O}^i(x) \ , \tag{2.65}$$

where x denotes the different lattice sites and i the operators $\mathcal{O}^i(x)$. $g_i(x,a)$ are the couplings (or sources) of the operators at point x of the lattice. The coupling constant $g_i(x,a)$ also depends on the lattice spacing a. In the renormalization group approach we coarse-grain the lattice by increasing the lattice spacing a and by replacing multiple sites by a single site with the average value of the lattice variables. During this process the Hamiltonian H stays in its form (2.65) but the different operators are weighed differently. This results in a change of the couplings $g_i(x, a)$ in each step. If we increase the lattice spacing a in a certain way where $u = (a_1, a_2, \dots)$ is the length scale at which we probe the system with $a_{n+1} > a_n$, then the couplings acquire a dependence of the scale (the lattice spacing a) and we can write $J_i(x, u)$. The evolution of the couplings in relation to the scale is determined by the RG flow equations

$$u\frac{\partial}{\partial u}g_i(x, u) = \beta_i\Big(g_j(x, u), u\Big) \tag{2.66}$$

where β_i is the β-function of the i-th coupling constant. $g_j(x, u)$ is another set of couplings related to operators $\mathcal{O}^j(x)$ [37]. The gauge/gravity duality considers u as an extra dimension and thus the succession of lattices at different values of u are considered as layers of a new higher-dimensional spacetime. Furthermore the sources/couplings $g_i(x, u)$ are regarded as fields $\phi_i(x, u)$ in a spacetime with one extra dimension. In the case of higher spin gravity the coupling g of the CFT is identified with the free 0-form field $C(x|Y)$ of the HS equations. $B(x|Y, Z)$ is getting $C(x|Y)$ if $B(x|Y, Z)$ is not any more dependent of Z (see chapter 2.2 and later 3.4). The dynamics of the sources/couplings $\phi_i(x, u)$ are governed by some action and according to gauge/gravity duality are also determined by some gravity theory, i.e. by a metric. Therefore, we can consider the gauge/gravity duality as a geometrization of quantum dynamics encoded by the renormalization group. The couplings in the UV where the lattice spacing a and thus u is small can be identified with the values of the bulk fields at the boundary of the $d+1$-dimensional spacetime. This boundary is called conformal boundary and lays at $z = 0$ for the AdS spacetime or future boundary at $\tau = 0$ for the dS spacetime (see chapter 2.1). We can state that the field theory lives on the boundary of the higher-dimensional spacetime [37].

Within the exact renormalization group (ERG) also irrelevant couplings are taken into account. We will use ERG and RG synonymously. RG is an auxiliary construction in quantum field theory. This means that the physics are contained in the partition function $\mathcal{Z}$, which are coupled to operators $\mathcal{O}^i$ via various sources/couplings g_i, and RG is just one way of extracting the physics [47]. A fundamental requirement of the ERG is that the partition function $\mathcal{Z}$ is left invariant under the RG flow, as it was the case for the Hamiltonian (2.65). The partition function $\mathcal{Z}$ reads

$$\mathcal{Z} = \int\limits_{p^2 \leq \Lambda^2} \mathcal{D}\phi e^{-S_\Lambda[\phi]} \tag{2.67}$$

where S_Λ is the Wilsonian effective action which is a central ingredient of ERG. If we have modelled a system by providing a description at a high energy scale, the bare scale Λ_0, then this description is provided by the bare action S_{Λ_0} which encodes the types and strengths of the various interactions. Now we coarse-grain our system, which in momentum space corresponds to integrating out degrees of freedom with high momenta between the bare scale and a lower, effective scale Λ. In general the action S will change during this procedure and result in a Wilsonian effective action S_Λ. We consider S_Λ to provide the appropriate description of physics at the effective scale. For reasons of simplicity we assume a single scalar field $\phi(x)$ [47]. Thus the partition function $\mathcal{Z}$ is linked via S_Λ and consequently via some Lagrangian $\mathcal{L}$ to the Hamiltonian H which for its part depends on the coupling g. Therefore on the left-hand side of (2.64) we can in principle replace g by $\exp(S_\Lambda[\phi])$ as well as the length scale u by a corresponding cut-off Λ with $u \sim \frac{1}{\Lambda}$ to get

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{-S_\Lambda[\phi]} = \dots \ . \tag{2.68}$$

At this place we would like to state that there are many varieties of essentially equivalent RG flow equations (e.g. [46, 48, 49]).

We will relate the right-hand side of (2.64), $\beta(g(u))$, to an object Ψ which parametrizes the continuum analogue of the Kadanoff blocking. Ψ is in general both a function and a functional of the field ϕ. Ψ corresponds to a (continuum) blocking procedure, where the blocking is performed only over local patches, and Ψ ensures UV regularization of the flow equation, which is achieved by including a (suitable) strong UV cut-off in Ψ [47]. In the blocking procedure the effective field $\phi(x)$ is written as some average over the bare field $\phi_0(x)$ as $\phi(x) = b_\Lambda[\phi_0](x)$. We can write the effective action S_Λ in terms of the bare action S_{Λ_0} by using the blocking functional

$$e^{-S_\Lambda[\phi]} = \int \mathcal{D}\phi_0 \ \delta\Big[\phi - b_\Lambda[\phi_0]\Big] e^{-S_{\Lambda_0}[\phi_0]} \tag{2.69}$$

which leaves the partition function $\mathcal{Z}$ invariant. We now define Ψ_Λ as follows [50]

$$\Psi_\Lambda(x) = e^{S_\Lambda[\phi]} \int \mathcal{D}\phi_0 \ \delta\Big[\phi - b_\Lambda[\phi_0]\Big] \Lambda \frac{\partial b_\Lambda[\phi_0](x)}{\partial \Lambda} e^{-S_{\Lambda_0}[\phi_0]} \ . \tag{2.70}$$

With definition (2.70) the blocking functional (2.69) can be described according to the RG flow equation[9]

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{-S_\Lambda[\phi]} = \int_x \frac{\delta}{\delta\phi(x)} \Big(\Psi_\Lambda(x) e^{-S_\Lambda[\phi]} \Big) \tag{2.71}$$

with $\frac{\delta\phi(x)}{\delta\phi(y)} = \delta^d(x-y)$ and $\delta F[\phi] = \int \delta\phi \frac{\delta}{\delta\phi} F[\phi]$. The RG flow equation (2.71) ensures that the basic functional integral defining the quantum field theory for the effective action S_Λ is invariant under changes in the cut-off Λ [44]. Performing the differentiation in the l.h.s and the functional derivative on the r.h.s of (2.71) we get the equivalent RG equation

$$-\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda[\phi] = \int_x \Psi_\Lambda(x) \frac{\delta S_\Lambda[\phi]}{\delta\phi(x)} - \int_x \frac{\delta \Psi_\Lambda(x)}{\delta\phi(x)} \ . \tag{2.72}$$

The two terms on the right-hand side are called the classical and quantum terms, respectively, as the first term generates tree-like diagrams whereas the second generates loop diagrams. However, it should be noted that the classical diagrams also have vertices which incorporate quantum fluctuations down to the effective scale, so the name 'classical' should not be interpreted to strictly.

The cut-off Λ sets the fundamental scale and we reduce the equation (2.72) to a dimensionless form by requiring [44]

$$\phi(x) := \Lambda^\delta \varphi(x\Lambda) \quad , \quad \frac{\delta}{\delta\phi(x)} := \Lambda^{d-\delta} \frac{\delta}{\delta\varphi(x\Lambda)} \quad , \quad \Psi_\Lambda(x) := \Lambda^\delta \Psi_t(x\Lambda) \tag{2.73}$$

[9] Definitions and conventions of dot-product, integrals and functionals are shown in appendix A.

where δ is the canonical dimension. Using $t = -\ln(\Lambda/\Lambda_0)$ and $S_\Lambda[\phi] = S_t[\varphi]$ the RG equation (2.72) becomes

$$\left(\frac{\partial}{\partial t} + D\varphi \cdot \frac{\delta}{\delta\varphi}\right) S_t[\varphi] = \Psi_t[\varphi] \cdot \frac{\delta}{\delta\varphi} S_t[\varphi] - \frac{\delta}{\delta\varphi} \cdot \Psi_t[\varphi] \ , \tag{2.74}$$

where we introduced the Dilatation operator D as[10]

$$D\varphi(x) = (x\partial_x + \delta)\varphi(x) \tag{2.75}$$

and the dot-product

$$\phi \cdot \psi = \psi \cdot \phi = \int d^d x \phi(x)\psi(x) \ . \tag{2.76}$$

The class of possible $\Psi_t[\varphi]$ satisfying the above conditions is not clear yet [44]. One way to simplify (2.74) is to choose

$$\Psi_t[\varphi] = \frac{1}{2} \ G \cdot \frac{\delta}{\delta\varphi} S_t[\varphi] \ , \tag{2.77}$$

and we finally get

$$\left(\frac{\partial}{\partial t} + D\varphi \cdot \frac{\delta}{\delta\varphi}\right) \mathcal{S}_t[\varphi] = \frac{1}{2} \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi] \cdot G \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi] - \frac{1}{2} \frac{\delta}{\delta\varphi} \cdot G \cdot \frac{\delta}{\delta\varphi} \mathcal{S}_t[\varphi] \ . \tag{2.78}$$

G is the 'ERG kernel' which is quasi-local and incorporates the UV regularization. Through the equation $G(p^2) = 2\mathcal{K}'(p^2)$ is entering the UV-regularized propagator $\mathcal{K}(p)/p^2$ with $\mathcal{K}(0) = 1$ and $\mathcal{K}(p^2 \to \infty) = 0$. For this, we have to get to momentum space, therefore we perform a Fourier transformation as

$$\begin{aligned}
\phi \cdot \psi = \int d^d x \phi(x)\psi(x) &= \int d^d x \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} e^{ipx} e^{iqx} \tilde{\phi}(p)\tilde{\psi}(q) \\
&= \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \left(\int d^d x e^{i(p+q)x}\right) \tilde{\phi}(p)\tilde{\psi}(q) = \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} (2\pi)^d \delta(p+q)\tilde{\phi}(p)\tilde{\psi}(q) \\
&= \int \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p)\tilde{\psi}(-p) \equiv \int_p \phi(p)\psi(-p) \ .
\end{aligned} \tag{2.79}$$

For the dot-product of field and variation, due to $\frac{\delta}{\delta\psi(x)} = \int \frac{d^d p}{(2\pi)^d} e^{-ipx} \frac{\delta}{\delta\psi(p)}$ [47] we find

$$\phi \cdot \frac{\delta}{\delta\psi} = \int_p \phi(p) \frac{\delta}{\delta\psi(p)} \ . \tag{2.80}$$

[10] We use $x \cdot \partial_x \equiv x\partial_x$ as we introduce a new dot-product in (2.76).

3 Relating renormalization group equations of the linear O(N) model to higher spin gravity in Anti-de Sitter spacetime

In this chapter we relate the renormalization group equations of the linear O(N) model to the higher spin gravity in Anti-de Sitter spacetime in 4 dimensions (AdS_4). In doing so, we follow the discussion of [26] and describe its calculations in detail.[11] In the first subchapter we will present the RG equation of the linear O(N) model and calculate some examples. In the subchapter 3.2 we then introduce the conformal operators expressed by the oscillators y^+_α and y^-_α in order to arrive at the higher spin equations. In 3.3 we then express the linearized RG flow as HS equations on AdS. The last two subchapters deal with gauge transformations and the inclusion of the inhomogenity.

3.1 Renormalization group equation of the linear O(N) model

Starting point of the relation between the renormalization group equations and the higher spin equations is the Polchinski equation for a single scalar field in $d = 2 + 1$ dimensions, with conventions of [44, 47] (see appendix A and chapter 2.4) and expressed in terms of dimensionless coordinates, momenta and fields

$$\left(\frac{\partial}{\partial t} + D\varphi \cdot \frac{\delta}{\delta\varphi}\right) \mathcal{S}_t[\varphi] = \frac{1}{2}\frac{\delta}{\delta\varphi}\mathcal{S}_t[\varphi] \cdot G \cdot \frac{\delta}{\delta\varphi}\mathcal{S}_t[\varphi] - \frac{1}{2}\frac{\delta}{\delta\varphi} \cdot G \cdot \frac{\delta}{\delta\varphi}\mathcal{S}_t[\varphi] \ . \tag{3.1}$$

D is the dilatation operator and in momentum space acts on the field[12] $\varphi(p)$ (with canonical dimension δ) as

$$D\varphi(p) = \Big(p\partial_p + [\varphi(p)]\Big)\varphi(p) = -(p\partial_p + d - \delta)\varphi(p) \ . \tag{3.2}$$

We can extract the renormalization group equations by expanding $\mathcal{S}_t$ in the fields and couplings by setting r.h.s. of (3.1) to zero due to linearized approximation for a quadratic action and neglected cosmological constant Λ_0.

We now want to study the RG equations for actions $\mathcal{S}_t$ with general couplings $g^{\mu_1 \cdots \mu_n}(t)$. Before doing so we consider two examples, the kinetic term and the mass-term of the O(N) model. In other words, the action $\mathcal{S}_t[\varphi]$ is given by the Lagrangian

$$\mathcal{L}_t[\varphi] = -\frac{1}{2}g^{\mu\nu}(t)\partial_\mu\varphi(x)\partial_\nu\varphi(x) - \frac{1}{2}m(t)^2\varphi(x)^2 \ . \tag{3.3}$$

Compared to (2.31) we replaced $\eta^{\mu\nu}$ with $g^{\mu\nu}(t)$ as well as ϕ with the rescaled fields φ as shown in 2.4. We note that the couplings $g^{\mu_1 \cdots \mu_n}(t)$ and consequently the mass $m(t)$ and the metric $g^{\mu\nu}(t)$ all depend on the renormalization time

$$t = -\ln(\Lambda/\Lambda_0) \ . \tag{3.4}$$

[11] Some typos were corrected.

[12] In order to simplify notation, we omit the tilde of φ, i.e. $\tilde{\varphi}(p) \equiv \varphi(p)$. If not explicitly stated, the field φ is in momentum space.

In the first example we use the mass-term $-\frac{1}{2}m(t)^2\varphi(x)^2$ of the O(N) model, whose action according to (2.79) in momentum space reads

$$\mathcal{S}_t = -\frac{1}{2}m(t)^2 \int \varphi(p)\varphi(-p) \tag{3.5}$$

where the coupling is $m(t)$ and we start with

$$-\left(\frac{\partial}{\partial t} + D\varphi \cdot \frac{\delta}{\delta\varphi}\right)\frac{m(t)^2}{2}\int \varphi(p)\varphi(-p) = 0 \; .$$

Due to

$$\begin{aligned}
-\left(\frac{\partial}{\partial t} + D\varphi \cdot \frac{\delta}{\delta\varphi}\right)\frac{m(t)^2}{2}\int\varphi(p)\varphi(-p) &= -\partial_t\frac{m(t)^2}{2}\int\varphi(p)\varphi(-p) - \frac{m(t)^2}{2}D\varphi\cdot\frac{\delta}{\delta\varphi}\int\varphi(p)\varphi(-p) \\
&= -\partial_t\frac{m(t)^2}{2}\int\varphi(p)\varphi(-p) - \frac{m(t)^2}{2}\int\frac{d^dq}{(2\pi)^d}D\varphi(q)\frac{\delta}{\delta\varphi(q)}\int\frac{d^dp}{(2\pi)^d}\varphi(p)\varphi(-p) \\
&= -\partial_t\frac{m(t)^2}{2}\int\varphi(p)\varphi(-p) \\
&\quad -\frac{m(t)^2}{2}\int\frac{d^dq}{(2\pi)^d}\int\frac{d^dp}{(2\pi)^d}D\varphi(q)\Big((2\pi)^d\delta(q-p)\varphi(-p) + (2\pi)^d\delta(q+p)\varphi(p)\Big) \\
&= -\partial_t\frac{m(t)^2}{2}\int\varphi(p)\varphi(-p) - \frac{m(t)^2}{2}\int\frac{d^dp}{(2\pi)^d}D\Big(\varphi(p)\varphi(-p) + \varphi(-p)\varphi(p)\Big) \\
&= -\partial_t\frac{m(t)^2}{2}\int\varphi(p)\varphi(-p) + m(t)^2\int(p\partial_p + d - \delta)\Big(\varphi(p)\varphi(-p)\Big) \\
&= \Big(-\partial_t\frac{m(t)^2}{2} + m(t)^2(d-\delta)\Big)\int\varphi(p)\varphi(-p) + m(t)^2\int\Big(p\partial_p\varphi(p)\varphi(-p) + \varphi(p)p\partial_p\varphi(-p)\Big)
\end{aligned}$$

we find

$$\Big(-\partial_t\frac{m(t)^2}{2} + m(t)^2(d-\delta)\Big)\int\varphi(p)\varphi(-p) + m(t)^2\int\Big(p\partial_p\varphi(p)\varphi(-p) + \varphi(p)p\partial_p\varphi(-p)\Big) = 0 \; . \tag{3.6}$$

Partially integrating the last term leads to

$$\begin{aligned}
&\Big(-\partial_t\frac{m(t)^2}{2} + m(t)^2(d-\delta)\Big)\int\varphi(p)\varphi(-p) + m(t)^2\int p\partial_p\varphi(p)\varphi(-p) \\
&\qquad + m(t)^2\varphi(p)p\varphi(-p)|_{-\infty}^{\infty} - m(t)^2\int\varphi(p)\varphi(-p) - m(t)^2\int p\partial_p\varphi(p)\varphi(-p) = 0 \; .
\end{aligned}$$

The boundary term vanishes and hence we then obtain

$$\Big(-\frac{\partial_t m(t)^2}{2} + m(t)^2(d-\delta-1)\Big)\int\varphi(p)\varphi(-p) = 0 \; .$$

With $\delta = 2\ \frac{d-2}{2} = d-2$ we get

$$\Big(-\frac{\partial_t m(t)^2}{2} + m(t)^2\Big)\int\varphi(p)\varphi(-p) = 0$$

and as $\int\varphi(p)\varphi(-p)$ is not zero we finally have

$$-\frac{\partial_t m(t)^2}{2} + m(t)^2 = 0 \quad \text{or} \quad \partial_t m(t)^2 - 2m(t)^2 = 0 \; . \tag{3.7}$$

Solving (3.7) we get

$$m(t)^2 = m_0^2 e^{2t} = m_0^2\left(\frac{\Lambda_0}{\Lambda}\right) \; , \tag{3.8}$$

which we expected for the scaling behaviour from dimensional analysis.

As a second example we take the action of the kinetic term

$$\mathcal{S}_t = -\frac{1}{2} g^{\mu\nu}(t) \int p_\mu \varphi(p) p_\nu \varphi(-p) \tag{3.9}$$

of the O(N) model in momentum space. This means we now use the new coupling $g^{\mu\nu}(t)$, which is the metric and our starting point is

$$-\left(\frac{\partial}{\partial t} + D\varphi \cdot \frac{\delta}{\delta\varphi}\right) \frac{1}{2} g^{\mu\nu}(t) \int p_\mu \varphi(p) p_\nu \varphi(-p) = 0 \ .$$

An analogous calculation as for (3.6) leads to

$$\left(-\frac{\partial_t g^{\mu\nu}(t)}{2} + g^{\mu\nu}(t)(d-\delta)\right) \int p_\mu \varphi(p) p_\nu \varphi(-p) + g^{\mu\nu}(t) \int \Big(p_\mu p \partial_p \varphi(p) p_\nu \varphi(-p) + p_\mu \varphi(p) p_\nu p \partial_p \varphi(-p)\Big) = 0 \ . \tag{3.10}$$

With partial integration as above we get

$$\Big(-\frac{\partial_t g^{\mu\nu}(t)}{2} + g^{\mu\nu}(t)(d-\delta)\Big) \int p_\mu \varphi(p) p_\nu \varphi(-p) + g^{\mu\nu}(t) \int p_\mu p \partial_p \varphi(p) p_\nu \varphi(-p) + g^{\mu\nu}(t) p_\mu \varphi(p) p_\nu p \varphi(-p)|_{-\infty}^{\infty}$$

$$-g^{\mu\nu}(t) \int p_\mu \varphi(p) p_\nu \varphi(-p) - g^{\mu\nu}(t) \int p_\mu p \partial_p \varphi(p) p_\nu \varphi(-p) = 0$$

which leads to

$$\Big(-\frac{\partial_t g^{\mu\nu}(t)}{2} + g^{\mu\nu}(t)(d-\delta-1)\Big) \int p_\mu \varphi(p) p_\nu \varphi(-p) = 0$$

and with $\delta = d - 1 = 2\ \frac{d-1}{2}$ to

$$-\frac{1}{2} \partial_t g^{\mu\nu}(t) \int p_\mu \varphi(p) p_\nu \varphi(-p) = 0 \ .$$

As $\int p_\mu \varphi(p) p_b \varphi(-p) \neq 0$ we finally find

$$\partial_t g^{\mu\nu}(t) = 0$$

and see that the metric does not depend on the renormalization time t and hence the cut-off Λ.

Now we perform the calculation for the general case with couplings $g^{\mu_1 \cdots \mu_n}(t)$. In this case for interactions of the form

$$\mathcal{S}_t = g^{\mu_1 \cdots \mu_n}(t) \int p_{\mu_1} \cdots p_{\mu_n} \varphi(p) \varphi(-p) \tag{3.11}$$

we start with

$$\left(\frac{\partial}{\partial t} + D\varphi \cdot \frac{\delta}{\delta\varphi}\right) g^{\mu_1 \cdots \mu_n}(t) \int p_{\mu_1} \cdots p_{\mu_n} \varphi(p) \varphi(-p) = 0$$

and performing the same steps as for m^2 and $g^{\mu\nu} p_\mu p_\nu$ we get

$$\Big(\partial_t g^{\mu_1 \cdots \mu_n}(t) - g^{\mu_1 \cdots \mu_n}(t)(2d - 2\delta - 2)\Big) \int p_{\mu_1} \cdots p_{\mu_n} \varphi(p) \varphi(-p) = 0$$

and with $\delta = 2\ \frac{d-2+n/2}{2}$ this equals

$$\Big(\partial_t g^{\mu_1 \cdots \mu_n}(t) + g^{\mu_1 \cdots \mu_n}(t)(n-2)\Big) \int p_{\mu_1} \cdots p_{\mu_n} \varphi(p) \varphi(-p) = 0 \ .$$

This leads to the important result of the Polchinski equation (3.1) for general couplings $g^{\mu_1\cdots\mu_n}(t)$ at linear order

$$\Big(\partial_t g^{\mu_1\cdots\mu_n}(t) + g^{\mu_1\cdots\mu_n}(t)(n-2)\Big)\frac{\partial}{\partial g^{\mu_1\cdots\mu_n}(t)}\mathcal{S}_t[g_t,\varphi] = 0\ . \tag{3.12}$$

This is only valid if $\mathcal{S}_t$ is quadratic in φ, thus if n is even, as for n odd

$$\begin{aligned} D\varphi\cdot\frac{\delta}{\delta\varphi}\mathcal{S}_t[\varphi] = \frac{1}{(2\pi)^d}\Big[\int d^dp D\varphi(p)p_{\mu_1}\cdots p_{\mu_n}g^{\mu_1\cdots\mu_n}(t)\varphi(-p) \\ + (-1)^n\int d^dp D\varphi(p)p_{\mu_1}\cdots p_{\mu_n}g^{\mu_1\cdots\mu_n}(t)\varphi(-p)\Big] \\ = 0\ . \end{aligned}$$

Thus for n being odd

$$\begin{aligned} \left(\frac{\partial}{\partial t} + D\varphi\cdot\frac{\delta}{\delta\varphi}\right)\mathcal{S}_t = \partial_t\mathcal{S}_t &\overset{(3.12)}{=} \Big(\partial_t g^{\mu_1\cdots\mu_n}(t) + g^{\mu_1\cdots\mu_n}(t)(n-2)\Big)\frac{\partial}{\partial g^{\mu_1\cdots\mu_n}(t)}\mathcal{S}_t \\ &= \partial_t\mathcal{S}_t + g^{\mu_1\cdots\mu_n}(t)(n-2)\frac{\partial}{\partial g^{\mu_1\cdots\mu_n}(t)}\mathcal{S}_t \end{aligned}$$

with $g^{\mu_1\cdots\mu_n}(t)\neq 0$ this means

$$\frac{\partial}{\partial g^{\mu_1\cdots\mu_n}(t)}\mathcal{S}_t = 0\ .$$

Thus $\mathcal{S}_t$ does not depend on $g^{\mu_1\cdots\mu_n}(t)$ for n being odd.

Finally, as a last example, in the φ^6-theory with $2+1$ dimensions the marginal deformation is

$$\mathcal{S}_t = \lambda\int\varphi(p_1)\cdots\varphi(p_6)\delta^3(p_1+\cdots+p_6)$$

where

$$\begin{aligned} D\varphi\cdot\frac{\delta}{\delta\varphi}\mathcal{S}_t &= \int\frac{d^dq}{(2\pi)^d}D\varphi(q)\frac{\delta}{\delta\varphi(q)}\mathcal{S}_t \\ &= \int\frac{d^dq}{(2\pi)^d}D\varphi(q)\frac{\delta}{\delta\varphi(q)}\lambda\int\frac{d^dp}{(2\pi)^d}\varphi(p_1)\cdots\varphi(p_6)\delta^3(p_1+\cdots+p_6) \\ &= \lambda\int\frac{d^dq}{(2\pi)^d}\int\frac{d^dp}{(2\pi)^d}D\varphi(q)\Big((2\pi)^d\delta(q-p_1)\varphi(p_2)\cdots\varphi(p_6) \\ &\qquad + \varphi(p_1)(2\pi)^d\delta(q-p_2)\varphi(p_3)\cdots\varphi(p_6)+\ldots\Big)\delta^3(p_1+\cdots+p_6) \\ &= 6\lambda\int\frac{d^dp}{(2\pi)^d}D\varphi(p_1)\cdots\varphi(p_6)\delta^3(p_1+\cdots+p_6) \\ &= -6\lambda\int\frac{d^dp}{(2\pi)^d}(p\partial_p + d - \delta)\varphi(p_1)\cdots\varphi(p_6)\delta^3(p_1+\cdots+p_6)\ . \end{aligned}$$

By using $p = p_1+\cdots+p_6$ and integrating out the first term it vanishes and we stay with

$$D\varphi\cdot\frac{\delta}{\delta\varphi}\mathcal{S}_t = -(d-\delta)\ 6\lambda\int\frac{d^dp}{(2\pi)^d}\varphi(p_1)\ldots\varphi(p_6)\delta^3(p_1+\cdots+p_6) = -6(d-\delta)\mathcal{S}_t\ ,$$

which with $\delta = 6\ \frac{d-2}{2}$ leads to

$$D\varphi\cdot\frac{\delta}{\delta\varphi}\mathcal{S}_t = -6\left(d - 6\ \frac{d-2}{2}\right)\mathcal{S}_t = 12(d-3)\mathcal{S}_t\ .$$

We recognize that for $d=3$ the term vanishes.

3.2 Towards the higher spin equations

We will now look at the renormalization group flow of the traceless subset of general couplings of the form (3.11), starting with a linearized and classical approximation. For this case we will show that the RG-equation (3.1) is a free higher spin equation on AdS_4. According to [40], the dilatation operator D, which is part of the conformal algebra in $2+1$ dimensions, has a Weyl star-product realization in terms of quadratic products of $(y^-_\alpha, y^{+\alpha})$ with $[y^-_\alpha, y^{+\beta}]_* = \delta^\beta_\alpha$ (compare chapter 2.2). y^+_α and y^-_α are oscillators made of the two-component spinors (twistors) y_α and $\bar{y}_\alpha$ in the following way [40][13]:

$$y^+_\alpha = \frac{1}{2}(y_\alpha - i\bar{y}_\alpha)\ ,\ y^-_\alpha = \frac{1}{2}(\bar{y}_\alpha - iy_\alpha)\ \text{ and thus }\ y_\alpha = y^+_\alpha + iy^-_\alpha\ ,\ \bar{y}_\alpha = y^-_\alpha + iy^+_\alpha\ . \tag{3.13}$$

The dilatation operator D, the translation generator $P_{\alpha\beta}$, the Lorentz transformations $L^\alpha{}_\beta$ and the special conformal transformation $K_{\alpha\beta}$ are

$$D = \frac{1}{2}y^{+\alpha}y^-_\alpha\ , \qquad P_{\alpha\beta} = iy^-_\alpha y^-_\beta\ , \tag{3.14}$$

$$L^\alpha{}_\beta = y^{+\alpha}y^-_\beta - \frac{1}{2}\delta^\alpha_\beta y^{+\gamma}y^-_\gamma\ , \qquad K_{\alpha\beta} = -iy^+_\alpha y^+_\beta\ . \tag{3.15}$$

We will use the Weyl (Moyal) star-product and its commutator relations as introduced in chapter 2.2.

In order to get familiar with the conformal algebra we calculate the 9 commutation relations of the 4 generators. In order to use the introduced star product, the generators must be rewritten with the two-component spinors (twistors) y_α and $\bar{y}_\alpha$ instead of the oscillators y^+_α and y^-_α, as

$$\begin{aligned}
D &= \frac{1}{2}y^{+\alpha}y^-_\alpha = -\frac{1}{4}y_\alpha\bar{y}^\alpha = -\frac{1}{4}\epsilon^{\alpha\beta}y_\alpha\bar{y}_\beta\ ,\\
P_{\alpha\beta} &= iy^-_\alpha y^-_\beta = \frac{i}{4}\Big(\bar{y}_\alpha\bar{y}_\beta - iy_\alpha\bar{y}_\beta - i\bar{y}_\alpha y_\beta - y_\alpha y_\beta\Big) = -K_{\alpha\beta}(y\leftrightarrow\bar{y})\ ,\\
L_{\alpha\beta} &= \epsilon_{\delta\alpha}L^\delta{}_\beta = \epsilon_{\delta\alpha}\Big(y^{+\delta}y^-_\beta - \frac{1}{2}\delta^\delta_\beta y^{+\gamma}y^-_\gamma\Big) = y^+_\alpha y^-_\beta - \frac{1}{4}\epsilon_{\delta\alpha}\delta^\delta_\beta y^\gamma\bar{y}_\gamma\\
&= \frac{1}{4}(y_\alpha - i\bar{y}_\alpha)(\bar{y}_\beta - iy_\beta) - \frac{1}{4}\epsilon_{\delta\alpha}\delta^\delta_\beta\epsilon^{\gamma\epsilon}y_\epsilon\bar{y}_\gamma\\
&= \frac{1}{4}(y_\alpha\bar{y}_\beta - i\bar{y}_\alpha\bar{y}_\beta - iy_\alpha y_\beta - \bar{y}_\alpha y_\beta) - \frac{1}{4}(\delta^\gamma_\beta\delta^\epsilon_\alpha - \delta^\epsilon_\beta\delta^\gamma_\alpha)y_\epsilon\bar{y}_\gamma = -\frac{i}{4}(\bar{y}_\alpha\bar{y}_\beta + y_\alpha y_\beta)\ ,\\
K_{\alpha\beta} &= -iy^+_\alpha y^+_\beta = -\frac{i}{4}\Big(y_\alpha y_\beta - i\bar{y}_\alpha y_\beta - iy_\alpha\bar{y}_\beta - \bar{y}_\alpha\bar{y}_\beta\Big)\ .
\end{aligned} \tag{3.16}$$

[13] In the 3D perspective we can omit the dots of the barred variables as they carry equivalent Lorentz representations. Later, when we transform from 3D to 4D, we will need again the dots.

To show a short example, the calculation of the first commutator is performed as

$$
\begin{aligned}
[D, P_\mu]_* = (\gamma_\mu)^{\gamma\delta}[D, P_{\gamma\delta}]_* &= -\frac{i}{16}(\gamma_\mu)^{\gamma\delta}\epsilon^{\alpha\beta}\Big[y_\alpha \bar{y}_\beta, \bar{y}_\gamma\bar{y}_\delta - iy_\gamma\bar{y}_\delta - i\bar{y}_\gamma y_\delta - y_\gamma y_\delta\Big]_* \\
&= -\frac{i}{8}(\gamma_\mu)^{\gamma\delta}\epsilon^{\alpha\beta}\Big(i\epsilon_{\beta\gamma}y_\alpha\bar{y}_\delta + i\epsilon_{\beta\delta}y_\alpha\bar{y}_\gamma + \epsilon_{\alpha\gamma}\bar{y}_\beta\bar{y}_\delta + \epsilon_{\beta\delta}y_\alpha y_\gamma \\
&\qquad + \epsilon_{\alpha\delta}\bar{y}_\beta\bar{y}_\gamma + \epsilon_{\beta\gamma}y_\alpha y_\delta - i\epsilon_{\alpha\gamma}\bar{y}_\beta y_\delta - i\epsilon_{\alpha\delta}\bar{y}_\beta y_\gamma\Big) \\
&= \frac{i}{4}(\gamma_\mu)^{\gamma\delta}\Big(y_\gamma y_\delta - \bar{y}_\gamma\bar{y}_\delta + i\bar{y}_\gamma y_\delta + iy_\gamma\bar{y}_\delta\Big) = -(\gamma_\mu)^{\gamma\delta}P_{\gamma\delta} = -P_\mu \ .
\end{aligned}
$$

The detailed calculations of all commutators can be found in C.2 and yield the following results, which are forming a conformal algebra:

$$
\begin{aligned}
[D, P_\mu]_* &= -P_\mu \\
[D, K_\mu]_* &= K_\mu \\
[K_\mu, P_n]_* &= 2\eta_{\mu\nu}D + 2L_{\mu\nu} \\
[L_{\mu\nu}, L_{\rho\sigma}]_* &= \eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\rho}L_{\mu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma} \\
[L_{nr}, K_\mu]_* &= \eta_{\rho\mu}K_\nu - \eta_{\nu\mu}K_\rho \\
[L_{nr}, P_\mu]_* &= \eta_{\rho\mu}P_\nu - \eta_{\nu\mu}P_\rho \\
[D, L_{\mu\nu}]_* &= 0 \\
[P_\mu, P_\rho]_* &= 0 \\
[K_\mu, K_\rho]_* &= 0 \ .
\end{aligned}
\tag{3.17}
$$

Next we replace $g^{\mu_1\cdots\mu_n}(t)p_{\mu_1}\ldots p_{\mu_n}$ by

$$
g^{(n)}(t) \equiv e^{-2t}g^{\mu_1\cdots\mu_n}(t)(\gamma_{\mu_1})^{\alpha_1\beta_1}\cdots(\gamma_{\mu_n})^{\alpha_n\beta_n}y^-_{\alpha_1}y^-_{\beta_1}\cdots y^-_{\alpha_n}y^-_{\beta_n} \ . \tag{3.18}
$$

Due to (3.14)

$$
(\gamma_\mu)^{\alpha\beta}y^-_\alpha y^-_\beta = -i(\gamma_\mu)^{\alpha\beta}P_{\alpha\beta} = -iP_\mu
$$

we get

$$
g^{(n)}(t) = e^{-2t}g^{\mu_1\cdots\mu_n}(t)(-1)^n(W_0)_{\mu_1}\cdots(W_0)_{\mu_n} = e^{-2t}g^{\mu_1\cdots\mu_n}(t)(-i)^nP_{\mu_1}\cdots P_{\mu_n} \ .
$$

Then twistor variables are used to represent the action p_μ of the dilatation operator D on $g^{\mu_1\cdots\mu_n}$.

With this the l.h.s. of (3.1) is equivalent to

$$
D_t g^{(n)}(t) \equiv \partial_t g^{(n)}(t) - [D, g^{(n)}(t)]_* \tag{3.19}
$$

which corresponds to a covariant derivative D_t of the coupling $g^{(n)}$, built of a standard derivative plus the connection D. The equivalence between the l.h.s. of (3.1) and the covariant derivative

D_t of the coupling (3.19) is shown in the following. Firstly, l.h.s. of (3.1) according to (3.12) corresponds to

$$\left(\partial_t + D\varphi \cdot \frac{\delta}{\delta\varphi}\right)\mathcal{S}_t[\varphi] = \left(\partial_t g^{\mu_1\cdots\mu_n}(t) + (n-2)g^{\mu_1\cdots\mu_n}(t)\right)e^{-2t}(-i)^n \int p_{\mu_1}\cdots p_{\mu_n}\varphi(p)\varphi(-p) \ . \tag{3.20}$$

Secondly, (3.19) developes into

$$\begin{aligned} D_t g^{(n)}(t) &\equiv \partial_t g^{(n)}(t) - [D, g^{(n)}(t)]_* \\ &= -2e^{-2t}g^{\mu_1\cdots\mu_n}(t)(-i)^n P_{\mu_1}\cdots P_{\mu_n} \\ &\quad + e^{-2t}\partial_t g^{\mu_1\cdots\mu_n}(t)(-i)^n P_{\mu_1}\cdots P_{\mu_n} - e^{-2t}g^{\mu_1\cdots\mu_n}(t)(-i)^n[D, P_{\mu_1}\cdots P_{\mu_n}]_* \end{aligned}$$

and by using $[D, P_{\mu_1}\cdots P_{\mu_n}]_* = -nP_{\mu_1}\cdots P_{\mu_n}$ (see appendix C.1) it follows that

$$D_t g^{(n)}(t) = \left(\partial_t g^{\mu_1\cdots\mu_n}(t) + (n-2)g^{\mu_1\cdots\mu_n}(t)\right)e^{-2t}(-i)^n P_{\mu_1}\cdots P_{\mu_n} \ ,$$

which is equivalent to (3.20) if

$$P_{\mu_1}\cdots P_{\mu_n} = \int p_{\mu_1}\cdots p_{\mu_n}\varphi(p)\varphi(-p) \ .$$

This means that the left-hand side of the Polchinski equation (3.1) is equivalent to the covariant derivative of the coupling $g^{(n)}$, including a standard derivative and the connection D.

Around the fixed point, for this class of couplings the linearized RG-equations thus become $D_t g^{(n)}(t) = 0$.

3.3 Linearized RG flow as HS equation on AdS

Still following [26] in this subchapter we will identify $D_t g^{(n)}(t) = 0$, i.e. the vanishing covariant derivative of the coupling, with the linearized higher spin equation of motion on AdS_4. Instead using oscillators y^+_α and y^-_α we express the generators through the two-component spinors (twistors) y_α and $\bar{y}_\alpha$ using relations (3.13). Doing so, the translation generator gets

$$P_{\alpha\beta} = iy^-_\alpha y^-_\beta = \frac{1}{4}(i\bar{y}_\alpha + y_\alpha)(\bar{y}_\beta - iy_\beta) = \frac{i}{4}\bar{y}_\alpha\bar{y}_\beta - \frac{i}{4}y_\alpha y_\beta + \frac{1}{4}\bar{y}_\alpha y_\beta + \frac{1}{4}y_\alpha\bar{y}_\beta \ .$$

We then transform $P_{\alpha\beta}$ to express it by 4D-spinor variables with left/right-handed Weyl spinors (twistors) with undotted/dotted indices and with $\epsilon_{\alpha\beta} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and ($\epsilon^{12} = \epsilon_{12} = 1$) (see appendix A)

$$P_{\alpha\beta} \mapsto \frac{i}{4}\bar{y}_{\dot\alpha}\bar{y}_{\dot\beta} - \frac{i}{4}y_\alpha y_\beta + \frac{1}{4}\bar{y}_{\dot\alpha}y_\beta + \frac{1}{4}y_\alpha\bar{y}_{\dot\beta} \ . \tag{3.21}$$

Expressed by the two-component spinors (twistors) the dilatation operator gets (calculation see appendix C.1)

$$D = \frac{1}{2}\epsilon_{\beta\alpha}y^{+\alpha}y^{-\beta} = \frac{1}{4}\epsilon_{\beta\alpha}y^\alpha\bar{y}^\beta \ .$$

Transforming D to express it with left/right-handed Weyl spinors (twistors) and using $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right) = -i\epsilon$ we get

$$D \mapsto -\frac{i}{4}(\sigma_2)_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} \ . \tag{3.22}$$

Now we compare (3.19) with the higher spin equations of [39]

$$dC + W * C - C * W = 0 \tag{3.23}$$

where $W = W_m dx^m$ is a higher spin gauge potential as a 1-form, thus a connection, and C is a field as a 0-form which will be identified with coupling constants describing perturbations of the RG-fixed point. The integrability conditions for (3.23) are

$$dW + W \wedge *W = 0 \ . \tag{3.24}$$

In the HS equations the operator d is the spacetime differential, with $d = dx^m(\frac{\partial}{\partial x^m})$.

The variant of the dilatation operator described in (3.22) adheres the AdS$_4$ solution to equation (3.23). Using the vielbein e_0 and the spin connection ω_0 the AdS$_4$ solution reads

$$W_0(x|Y) = e_0(x|Y) + \omega_0(x|Y) \tag{3.25}$$

where (in Poincaré coordinates and with curvature $L = 1$)

$$e_0(x|Y) = -\frac{1}{4i}\frac{dx^m}{z}(\sigma_m)_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} \ , \quad \omega_0(x|Y) = -\frac{1}{4i}\frac{dx^\mu}{z}((\sigma^{\mu z})_{\alpha\beta} y^\alpha y^\beta + (\bar{\sigma}^{\mu z})_{\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}}) \tag{3.26}$$

with the 4D sigma matrices $(\sigma_m)_{\alpha\dot{\beta}} = (1, \vec{\sigma})_{\alpha\dot{\beta}}$, $(\bar{\sigma}_m)^{\dot{\alpha}\beta} = (1, -\vec{\sigma})^{\dot{\alpha}\beta}$. m, n are the $d+1 = 4$ dimensional bulk indices with $m, n \in \{0, 1, 2, 3\}$. In (3.25), Y are the auxiliary spinor (twistor) variables while x describe spacetime coordinates (compare chapter 2.2 and [40]). Comparing (3.25) and (3.26) with the first line of the series expansion (2.14), we see that W_0 represents the spin 2 fields, thus representing a background metric. Higher spin fields are only introduced as fluctuation. In the next subchapter we will again consider the full higher spin fields when dealing with gauge transformations.

The 3D gamma matrices are obtained by deleting the matrix with space-time index $m = 2$, i.e. $(\gamma_\mu)_{\alpha\beta} = (1, \sigma_1, \sigma_3)_{\alpha\beta}$. Thus, in this subchapter and in appendix C.2 we use $\mu \in \{0, 1, 3\}$ in order to perform the 3D $\leftrightarrow$ 4D transformation. After entering e_0 and ω_0 into (3.25), W_0 should take the form $-\frac{dz}{z}D + i\frac{dx^\mu}{z}P_\mu$ (compare [28]), while using $(\gamma_\mu)^{\alpha\beta}P_{\alpha\beta} \equiv P_\mu$ and $x^2 = z$ and $\sigma^z = \sigma^2$, $\mu = 0, 1, 3$ ($m = 2$ was deleted). This is shown in the following (using conventions of appendix A and [40]), where we start with

$$W_0 = -\frac{1}{4i}\frac{dx^m}{z}(\sigma_m)_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} - \frac{1}{4i}\frac{dx^\mu}{z}((\sigma_{\mu z})_{\alpha\beta} y^\alpha y^\beta + (\bar{\sigma}_{\mu z})_{\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}})$$

$$\begin{aligned} = &-\frac{1}{4i}\frac{dx^0}{z}\delta_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} - \frac{1}{4i}\frac{dx^1}{z}(\sigma_1)_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} - \frac{1}{4i}\frac{dz}{z}(\sigma_2)_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} - \frac{1}{4i}\frac{dx^3}{z}(\sigma_3)_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} \\ &- \frac{1}{4i}\frac{dx^\mu}{z}\left[\left((\sigma_\mu)_{\alpha\dot{\alpha}}(\sigma_z)_\beta{}^{\dot{\alpha}} - (\sigma_z)_{\alpha\dot{\alpha}}(\sigma_\mu)_\beta{}^{\dot{\alpha}}\right) y^\alpha y^\beta + \left((\sigma_\mu)_{\alpha\dot{\alpha}}(\sigma_z)^\alpha{}_{\dot{\beta}} - (\sigma_z)_{\alpha\dot{\alpha}}(\sigma_\mu)^\alpha{}_{\dot{\beta}}\right) \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}}\right] . \end{aligned}$$

Then we use (3.22) as well as $(\sigma_z)_\beta{}^{\dot\alpha} = -i\epsilon_\beta{}^{\dot\alpha} = -i\delta^{\dot\alpha}_\beta$, $(\sigma_z)_{\alpha\dot\alpha} = -i\epsilon_{\alpha\dot\alpha}$ and $\epsilon_{\alpha\dot\alpha}\bar y^{\dot\alpha} = y_\alpha$ to receive

$$\begin{aligned} W_0 = &-\frac{dz}{z}D + \frac{i}{4}\frac{dx^\mu}{z}(\sigma_\mu)_{\alpha\dot\beta}y^\alpha\bar y^{\dot\beta} + \frac{i}{4}\frac{dx^\mu}{z}\Big(-i(\sigma_\mu)_{\alpha\dot\alpha}\delta^{\dot\alpha}_\beta + i\epsilon_{\alpha\dot\alpha}(\sigma_\mu)_\beta{}^{\dot\alpha}\Big)y^\alpha y^\beta \\ &+ \frac{i}{4}\frac{dx^\mu}{z}\Big(i(\sigma_\mu)_{\alpha\dot\alpha}\delta^\alpha_{\dot\beta} + i\epsilon_{\alpha\dot\alpha}(\sigma_\mu)^\alpha{}_{\dot\beta}\Big)\bar y^{\dot\alpha}\bar y^{\dot\beta} \\ = &-\frac{dz}{z}D + \frac{i}{4}\frac{dx^\mu}{z}(\sigma_\mu)_{\alpha\dot\beta}y^\alpha\bar y^{\dot\beta} + \frac{1}{4}\frac{dx^\mu}{z}(\sigma_\mu)_{\alpha\beta}y^\alpha y^\beta - \frac{1}{4}\frac{dx^\mu}{z}(\sigma_\mu)_{\dot\alpha\dot\beta}\bar y^{\dot\alpha}\bar y^{\dot\beta} \\ &- \frac{1}{4}\frac{dx^\mu}{z}\Big((\sigma_\mu)_\beta{}^{\dot\alpha}\bar y_{\dot\alpha}y^\beta - (\sigma_\mu)^\alpha{}_{\dot\beta}y_\alpha\bar y^{\dot\beta}\Big)\ . \end{aligned}$$

Now we move from the dotted/undotted 4D-spinors with Sigma-matrices (σ_m) to the undotted 3D-spinors with Gamma-matrices (γ_μ) using $i(\sigma_\mu)_\alpha{}^{\dot\beta} = (\gamma_\mu)_\alpha{}^{\dot\beta}$ and $i(\sigma_\mu)^\beta{}_{\dot\alpha} = (\gamma_\mu)_{\dot\alpha}{}^\beta$ (see appendix A) and get

$$\begin{aligned} W_0 = &-\frac{dz}{z}D + \frac{i}{4}\frac{dx^\mu}{z}(\gamma_a)_{\alpha\beta}y^\alpha\bar y^\beta + \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}y^\alpha y^\beta - \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}\bar y^\alpha\bar y^\beta \\ &+ \frac{i}{4}\frac{dx^\mu}{z}\Big((\gamma_\mu)_\beta{}^\alpha\bar y_\alpha y^\beta - (\gamma_\mu)_\beta{}^\alpha y_\alpha\bar y^\beta\Big) \\ = &-\frac{dz}{z}D + \frac{i}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}y^\alpha\bar y^\beta + \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}y^\alpha y^\beta - \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}\bar y^\alpha\bar y^\beta\ . \\ &+ \frac{i}{4}\frac{dx^\mu}{z}(\gamma_\mu)_\beta{}^\alpha\Big(\bar y_\alpha y^\beta - y_\alpha\bar y^\beta\Big)\ . \end{aligned}$$

Within the last term we use the identity $\bar y_\alpha y^\beta - y_\alpha\bar y^\beta = \bar y_\alpha y^\beta - \epsilon_{\gamma\alpha}y^\gamma\bar y^\beta = \bar y_\alpha y^\beta - \epsilon_{\gamma\alpha}y^\gamma\epsilon^{\beta\delta}\bar y_\delta$ $= \bar y_\alpha y^\beta - (\delta^\beta_\gamma\delta^\delta_\alpha - \delta^\beta_\alpha\delta^\delta_\gamma)y^\gamma\bar y_\delta = \bar y_\alpha y^\beta - y^\beta\bar y_\alpha + \delta^\beta_\alpha\bar y_\gamma y^\gamma = \delta^\beta_\alpha\bar y_\gamma y^\gamma = \delta^\beta_\alpha\epsilon_{\delta\gamma}\bar y^\delta y^\gamma$ and continue with

$$\begin{aligned} W_0 &= -\frac{dz}{z}D + \frac{i}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}y^\alpha\bar y^\beta + \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}y^\alpha y^\beta - \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}\bar y^\alpha\bar y^\beta + \frac{i}{4}\frac{dx^\mu}{z}(\gamma_\mu)_\beta{}^\alpha\delta^\beta_\alpha\epsilon_{\delta\gamma}\bar y^\delta y^\gamma \\ &= -\frac{dz}{z}D + \frac{i}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}y^\alpha\bar y^\beta + \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}y^\alpha y^\beta - \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}\bar y^\alpha\bar y^\beta + \frac{i}{4}\frac{dx^\mu}{z}(\gamma_\mu)_\alpha{}^\alpha\epsilon_{\delta\gamma}\bar y^\delta y^\gamma\ . \end{aligned}$$

Again in the last term, we set α to δ to obtain

$$W_0 = -\frac{dz}{z}D + \frac{i}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}y^\alpha\bar y^\beta + \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}y^\alpha y^\beta - \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}\bar y^\alpha\bar y^\beta + \frac{i}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\delta\gamma}\bar y^\delta y^\gamma$$

and then set δ, γ to α, β and get

$$\begin{aligned} W_0 &= -\frac{dz}{z}D + \frac{i}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}y^\alpha\bar y^\beta + \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}y^\alpha y^\beta - \frac{1}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}\bar y^\alpha\bar y^\beta + \frac{i}{4}\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}\bar y^\alpha y^\beta \\ &= -\frac{dz}{z}D + i\frac{dx^\mu}{z}(\gamma_\mu)_{\alpha\beta}P^{\alpha\beta} = -\frac{dz}{z}D + i\frac{dx^\mu}{z}P_\mu\ , \end{aligned}$$

and thus

$$W_0 = -\frac{dz}{z}D + i\frac{dx^\mu}{z}P_\mu\ . \tag{3.27}$$

Comparing this with

$$W_0 = (W_0)_t\frac{dz}{z} + (W_0)_\mu\frac{dx^\mu}{z} \tag{3.28}$$

and identifying $t = \ln z$ ($z = dz/dt$) we can read off for D and $P_\mu = (\gamma_\mu)^{\alpha\beta} P_{\alpha\beta}$ that

$$(W_0)_t = -D \quad \text{and} \quad (W_0)_\mu = iP_\mu \ . \tag{3.29}$$

The first identity of (3.29) further undermines the minus sign in (3.19).

Thus the tree-level (without loops), linearized renormalization group flow which is the left-hand side of (3.1) and is equivalent to (3.19) is identified with the higher spin equation

$$\partial_t C(x|Y) + [(W_0)_t(x|Y), C(x|Y)]_* = 0 \tag{3.30}$$

where the free field $C(x|Y)$ is identified with the coupling $g^{(n)}(t)$ as in (3.18) but with 4D-spinor variables. According to gauge/gravity duality a coupling in CFT corresponds to a field in the higher dimensional gravity theory (compare chap. 2.4). Having this in mind, (3.30) and the transformation from 3D to 4D implies

$$iP_\mu = i(\gamma_\mu)^{\alpha\beta} P_{\alpha\beta} = \frac{i}{4}(\sigma_\mu)^{\alpha\dot\beta} y_\alpha \bar y_{\dot\beta} + \frac{i}{4}\Big((\sigma_{\mu z})^{\alpha\beta} y_\alpha y_\beta + (\bar\sigma_{\mu z})^{\dot\alpha\dot\beta} \bar y_{\dot\alpha} \bar y_{\dot\beta}\Big)$$

and also (compare [39])

$$\begin{aligned} 4(\gamma_\mu)^{\alpha\beta} y^-_\alpha y^-_\beta &= -4i(\gamma_\mu)^{\alpha\beta} P_{\alpha\beta} \ (= -4iP_\mu) \\ &= -4\left[\frac{i}{4}(\sigma_\mu)^{\alpha\dot\beta} y_\alpha \bar y_{\dot\beta} + \frac{i}{4}\Big((\sigma_{\mu z}{}^{\alpha\beta} y_\alpha y_\beta + (\bar\sigma_{\mu z})^{\dot\alpha\dot\beta} \bar y_{\dot\alpha} \bar y_{\dot\beta}\Big)\right] \\ &= -i(\sigma_\mu)^{\alpha\dot\beta} y_\alpha \bar y_{\dot\beta} - i\Big((\sigma_{z\mu})^{\alpha\beta} y_\alpha y_\beta + (\bar\sigma_{z\mu})^{\dot\alpha\dot\beta} \bar y_{\dot\alpha} \bar y_{\dot\beta}\Big) \\ &= -4(W_0)_\mu \ . \end{aligned}$$

Furthermore with the aim to verify the other components of (3.23), which are

$$\partial_\mu C + [(W_0)_\mu, C]_* = 0 \tag{3.31}$$

we have seen in (3.29) that P_μ is proportional to $(W_0)_\mu$ in (3.25). This implies that (3.31) is satisfied as long as $C(x|Y)$ does not depend on x^μ, i.e. $C(x|Y) = C(t|Y)$.

According to [26], equation (3.30) is the tree-level linearized RG equation for the couplings $g^{(n)}$ around the Gaussian fixed point while (3.31) encodes translation invariance.

In the equations for $C(x|Y)$ in (3.30) and (3.31) is entering the commutator $[(W_0)_\mu, C]_*$ and not the twisted commutator $(W_0)_\mu * C - C * (\tilde W_0)_\mu$ with $\tilde f(x|y, \bar y) \equiv f(x|-y, \bar y)$. Thus $C(x|Y)$ is a non-propagating (auxiliary) field. This should be expected because the RG-equation is a first order rather than a second order equation. This means, the auxiliary sector is responsible for couplings (moduli) of the theory. A similar observation was made in 3D higher spin theory of matter fields of arbitrary mass, where the mass parameter is directly related to some value of the auxiliary fields.

To establish a complete relation between HS- and RG-equations we also consider the Vasiliev equations involving auxiliary spinor connection [39] (compare chapter 2.2)

$$S(Z, Y, k, \bar k) = s_\alpha dz^\alpha + s_{\dot\alpha} d\bar z^{\dot\alpha}$$

which are

$$\begin{aligned} S*S &= -idz_\alpha dz^\alpha(1+F(B)*\kappa) - id\bar{z}_{\dot\alpha}d\bar{z}^{\dot\alpha}(1+\bar{F}(B)*\bar\kappa) \\ S*B - B*S &= 0 \\ dS + W*S - S*W &= 0 \end{aligned} \tag{3.32}$$

with

$$Z = (z_\alpha, \bar{z}_{\dot\alpha}) \ ,$$

which is a pair of auxiliary twistor variables with commutation relations (2.18) and with

$$dz^\alpha d\bar{z}^{\dot\alpha} = -d\bar{z}^{\dot\alpha}dz^\alpha$$

as anticommuting differentials.

The field $B(x|Y,Z)$ follows $B(x|Y,Z)|_{Z=0} = C(x|Y)$ and the Vasiliev equations. $F(B)$ is an arbitrary function and $\kappa = kK$ where k is a Kleinian with

$$kf(z^\alpha, dz^\alpha, y^\alpha, z^{\dot\alpha}, dz^{\dot\alpha}, y^{\dot\alpha}) = f(-z^\alpha, -dz^\alpha, -y^\alpha, z^{\dot\alpha}, dz^{\dot\alpha}, y^{\dot\alpha})k \tag{3.33}$$

and $K = e^{iz_\alpha y^\alpha}$ an inner Kleinian for the $*$-product with

$$f(y,z)*K = f(-z,-y)K \quad , \quad K*f(y,z) = Kf(z,y) \ . \tag{3.34}$$

In the following we check (3.34), using the associativity of the $*$-product, $K*1 = K$, $K*K = 1$ and $y_\alpha z^\alpha = -z_\alpha y^\alpha$ (see appendix A and C.1):

$$f(y,z)*K = f(y,z)*e^{iz_\alpha y^\alpha} = (f(y,z)*e^{iz_\alpha y^\alpha})*(e^{iz_\beta y^\beta}e^{-iz_\beta y^\beta}) = f(y,z)e^{-iz_\beta y^\beta} = f(y,z)e^{iy_\alpha z^\alpha}$$

and setting y to $-z$ leads to

$$f(y,z)*K = f(-z,-y)e^{i(-z_\alpha)(-y^\alpha)} = f(-z,-y)K \ ,$$

while

$$K*f(y,z) = e^{iz_\alpha y^\alpha}*f(y,z) = (e^{-iz_\beta y^\beta}e^{iz_\beta y^\beta})*(e^{iz_\alpha y^\alpha}*f(y,z)) = e^{-iz_\beta y^\beta}f(y,z) = e^{iy_\alpha z^\alpha}f(y,z)$$

with $y = z$ leads to

$$K*f(y,z) = e^{iz_\alpha y^\alpha}f(z,y) = Kf(z,y) \ .$$

To 0-th order in B, first equation in (3.32) is solved by

$$S_0 = z_\alpha dz^\alpha + z_{\dot\alpha}dz^{\dot\alpha}$$

as can be seen below:

$$\begin{aligned} S_0*S_0 &= (z_\alpha dz^\alpha + z_{\dot\alpha}dz^{\dot\alpha})*(z_\beta dz^\beta + z_{\dot\beta}dz^{\dot\beta}) \\ &= (z_\alpha*z_\beta)dz^\alpha dz^\beta + (z_\alpha*z_{\dot\beta})dz^\alpha dz^{\dot\beta} + (z_{\dot\alpha}*z_\beta)dz^{\dot\alpha}dz^\beta + (z_{\dot\alpha}*z_{\dot\beta})dz^{\dot\alpha}dz^{\dot\beta} \ , \end{aligned}$$

now interchanging α and β in the third term

$$S_0 * S_0 = \frac{1}{2}(z_\alpha * z_\beta)dz^\alpha dz^\beta - \frac{1}{2}(z_\alpha * z_\beta)dz^\beta dz^\alpha + (z_\alpha * z_{\dot\beta})dz^\alpha dz^{\dot\beta} + (z_{\dot\beta} * z_\alpha)dz^{\dot\beta} dz^\alpha$$
$$+ \frac{1}{2}(z_{\dot\alpha} * z_{\dot\beta})dz^{\dot\alpha} dz^{\dot\beta} - \frac{1}{2}(z_{\dot\alpha} * z_{\dot\beta})dz^{\dot\beta} dz^{\dot\alpha} \ ,$$

and then by interchanging α and β in the second and last term, we get

$$S_0 * S_0 = \frac{1}{2}[z_\alpha, z_\beta]_* dz^\alpha dz^\beta + [z_\alpha, z_{\dot\beta}]_* dz^\alpha dz^{\dot\beta} + \frac{1}{2}[z_{\dot\alpha}, z_{\dot\beta}]_* dz^{\dot\alpha} dz^{\dot\beta}$$

and using the commutator relations for z_α (see chapter 2.2 and appendix A) we obtain the result

$$S_0 * S_0 = -i\epsilon_{\alpha\beta} dz^\alpha dz^\beta - i\epsilon_{\dot\alpha\dot\beta} dz^{\dot\alpha} dz^{\dot\beta} = -idz_\alpha dz^\alpha - idz_{\dot\alpha} dz^{\dot\alpha} \ .$$

With this, the second equation in (3.32) implies that B is independent of Z at leading order. To 0-th order in B the last equation is then again identically satisfied. On the other hand, for non-vanishing $F(B)$ this equation implies a correction to W.

3.4 Gauge transformations

While a complete understanding of how to represent arbitrary higher spin gauge transformations in renormalization group equations is still lacking, we can develop some intuition by looking at examples. In this chapter we again use the general coupling $g^{\mu_1\cdots\mu_n}$ and thus include all higher spin fields. According to chapter 2.2 the higher spin equations are

$$dB + W * B - B * W = 0 \quad , \quad dW + W \wedge *W = 0 \ , \tag{3.35}$$

while its gauge transformations have the following form:

$$\delta B(x|Y,Z) = [B(x|Y,Z), \epsilon(x|Y,Z)]_* \equiv [B,\epsilon]_* \tag{3.36}$$

$$\delta W(x|Y,Z) = d\epsilon(x|Y,Z) + [W(x|Y,Z), \epsilon(x|Y,Z)]_* \equiv d\epsilon + [W,\epsilon]_* \ . \tag{3.37}$$

Now we assume that $\epsilon = \epsilon(t|y^\alpha, \bar y^{\dot\alpha})$, then $B(x|Y,Z)$ gets $C(x|Y) = g^{(n)}(t)$ and (3.36) reduces to[14]

$$\delta g^{(n)} = [g^{(n)}, \epsilon]_* \ . \tag{3.38}$$

To get a 3D interpretation of $\delta g^{(n)}$ we express the 4D spinors y^α and $\bar y^{\dot\alpha}$ in terms of the 3D oscillators $y^{\pm\alpha}$. Taking $\epsilon = b^\mu(t)K_\mu$ with K_μ in 4D spinor variables, it amounts to

$$\delta g^{(n)} = -2ig^{(n)}(g\cdot b)^{(n-1)}D + 2i(g^{[\mu}b^{\nu]})^{(n-1)}L_{\mu\nu} \ . \tag{3.39}$$

For the proof of (3.39) we start with

$$\delta g^{(n)} = [g^{(n)}, \epsilon]_* = e^{-2t} g^{\mu_1\cdots\mu_n}(-i)^n b^\mu [P_{\mu_1}\cdots P_{\mu_n}, K_\mu]_*$$

[14] In order to simplify notation in the subchapters 3.4 and 3.5 we use $g^{(n)}(t) \equiv g^{(n)}$ and $g^{\mu_1\cdots\mu_n}(t) \equiv g^{\mu_1\cdots\mu_n}$, i.e. g still depends on the renormalization time t and hence the cut-off Λ.

and continue by using an identity for $[P_{\mu_1}\cdots P_{\mu_n}, K_\mu]_*$ shown in appendix C.1

$$\begin{aligned}\delta g^{(n)} &= e^{-2t} g^{\mu_1\dots\mu_n}(-i)^n b^\mu \Big([P_{\mu_1}, K_\mu]_* P_{\mu_2}\cdots P_{\mu_n} + P_{\mu_1}[P_{\mu_2}, K_\mu]_* P_{\mu_3}\cdots P_{\mu_n} + \cdots \Big) \\ &= e^{-2t} g^{\mu_1\dots\mu_n}(-i)^n b^\mu \Big(2(\eta_{\mu_1\mu} D - L_{\mu_1\mu}) P_{\mu_2}\cdots P_{\mu_n} + 2P_{\mu_1}(\eta_{\mu_2\mu} D - L_{\mu_2\mu}) P_{\mu_3}\cdots P_{\mu_n} + \cdots \Big) \\ &= 2e^{-2t} g^{\mu_1\dots\mu_n}\eta_{\mu_1\mu}(-i)^{n-1}(-i) b^\mu D P_{\mu_2}\cdots P_{\mu_n} + \dots \\ &\quad - e^{-2t} g^{\mu_1\mu_2\dots\mu_n}(-i)^{n-1}(-i) b^\mu 2 L_{\mu_1\mu} P_{\mu_2}\cdots P_{\mu_n} - \dots \ .\end{aligned}$$

In the latter term we set $\mu_1 \mapsto \mu$ and $\mu \mapsto \nu$ and get

$$\delta g^{(n)} = -2ie^{-2t} g_\mu{}^{\mu_2\dots\mu_n} b^\mu (-i)^{n-1} D P_{\mu_2}\cdots P_{\mu_n} - \dots + 2ie^{-2t} g^{\mu\mu_2\dots\mu_n} b^\nu (-i)^{n-1} L_{\mu\nu} P_{\mu_2}\cdots P_{\mu_n} + \dots$$

and then use $g_\mu{}^{\mu_2\cdots\mu_n} b^\mu = (g\cdot b)^{\mu_2\cdots\mu_n}$, $g^{\mu\mu_2\cdots\mu_n} b^\nu = (g^\mu b^\nu)^{\mu_2\cdots\mu_n}$ as well as the Weyl ordering to receive

$$\delta g^{(n)} = -2i(g\cdot b)^{(n-1)} D + 2i(g^{[\mu} b^{\nu]})^{(n-1)} L_{\mu\nu} \ .$$

This transformation does not leave $\mathcal{S}_t$ invariant in general, although some terms may vanish, e.g. for n being odd $\mathcal{S}_t$ does not depend on $g^{(n)}$ as already noted in chapter 3.1. Similarly, the Lorentz ($L_{\mu\nu}$) term on the r.h.s. of (3.39) vanishes in $\mathcal{S}_t$ for n being odd, while the dilatation (D) term does contribute a coupling with n being even.

The connection $D = -(W_0)_t$ transforms as

$$D \mapsto D + \delta D = D + d\epsilon + [D,\epsilon]_* = D + d(b^\mu(t) K_\mu) + [D, b^\mu(t) K_\mu]_* = D - \partial_t b^\mu K_\mu + b^\mu K_\mu \ , \quad (3.40)$$

where we used $d(b(t)) = dx^m \frac{\partial}{\partial x^m}(b(t)) = dx^0 \frac{\partial}{\partial x^0}(b(t)) = -dx_0 \frac{\partial}{\partial x^0}(b(t)) = -\partial_t(b(t))$.

It can be seen that $W_0 + \delta W$ is no longer in AdS-form. Nevertheless, (3.19) transforms convariantly by construction. With $\partial_t b \equiv \dot b$ we show that

$$\begin{aligned}[D,\delta g^{(n)}]_* + [\delta D, g^{(n)}]_* &= e^{-2t} g^{\mu_1\cdots\mu_n}(-i)^n b^\mu \Big[D, [P_{\mu_1}\cdots P_{\mu_n}, K_\mu]_* \Big]_* + [-\dot b^\mu K_\mu + b^\mu K_\mu, g^{(n)}]_* \\ &= e^{-2t} g^{\mu_1\cdots\mu_n}(-i)^n b^\mu \Big(- \Big[P_{\mu_1}\cdots P_{\mu_n}, [K_\mu, D]_* \Big]_* - \Big[K_\mu, [D, P_{\mu_1}\cdots P_{\mu_n}]_* \Big]_* \Big) \\ &\quad + [-\dot b^\mu K_\mu + b^\mu K_\mu, g^{(n)}]_*\end{aligned}$$

and by using $[K_\mu, D]_* = -K_\mu$ and $[D, P_{\mu_1}\dots P_{\mu_n}]_* = -nP_{\mu_1}\cdots P_{\mu_n}$ (see C.1) we get

$$\begin{aligned}[D,\delta g^{(n)}]_* + [\delta D, g^{(n)}]_* &= e^{-2t} g^{\mu_1\cdots\mu_n}(-i)^n b^\mu(-n+1)[P_{\mu_1}\cdots P_{\mu_n}, K_\mu]_* + [-\dot b^\mu K_\mu, g^{(n)}]_* + [b^\mu K_\mu, g^{(n)}]_* \\ &= (-n+1)\delta g^{(n)} - 2i(g\cdot \dot b)^{(n-1)} D + 2i(g^{[\mu}\dot b^{\nu]})^{(n-1)} L_{\mu\nu} + [\epsilon, g^{(n)}]_* \\ &= -n\delta g^{(n)} - 2i(g\cdot \dot b)^{(n-1)} D + 2i(g^{[\mu}\dot b^{\nu]})^{(n-1)} L_{\mu\nu} \qquad (3.41)\end{aligned}$$

as well as

$$\partial_t \delta g^{(n)} = -2i(\dot g\cdot b)^{(n-1)} D + 2i(\dot g^{[\mu} b^{\nu]})^{(n-1)} L_{\mu\nu} - 2i(g\cdot \dot b)^{(n-1)} D + 2i(g^{[\mu}\dot b^{\nu]})^{(n-1)} L_{\mu\nu} \ . \quad (3.42)$$

Now we will test the invariance of the transformation $\delta g^{(n)}$ on (3.19), i.e. if :

$$\partial_t(\delta g^{(n)}) - [D, \delta g^{(n)}]_* - [\delta D, g^{(n)}]_* = 0$$

is valid. This is the case, what can be seen in the following

$$\begin{aligned}\partial_t \delta g^{(n)} - [D, \delta g^{(n)}]_* - [\delta D, g^{(n)}]_* &= -2i(\dot{g} \cdot b)^{(n-1)} D + 2i(\dot{g}^{[\mu} b^{\nu]})^{(n-1)} L_{\mu\nu} + n\delta g^{(n)} \\ &= 2ni(g \cdot b)^{(n-1)} D - 2ni(g^{[\mu} b^{\nu]})^{(n-1)} L_{\mu\nu} \\ &\quad - 2ni(g \cdot b)^{(n-1)} D + 2ni(g^{[\mu} b^{\nu]})^{(n-1)} L_{\mu\nu} = 0 \ ,\end{aligned}$$

where we inserted

$$\dot{g}^{(n)} = -ng^{(n)} \tag{3.43}$$

which also leads to the fact that (3.19) is satisfied without using further properties of $\mathcal{S}_t$.

To show equation (3.43) we calculate

$$\begin{aligned}\dot{g}^{(n)} = \partial_t g^{(n)} &= \Big(e^{-2t}(-2) g^{\mu_1 \cdots \mu_n}(t) + e^{-2t} \partial_t g^{\mu_1 \cdots \mu_n}(t)\Big)(-i)^n P_{\mu_1} \dots P_{\mu_n} \\ &= \Big(- 2g^{\mu_1 \cdots \mu_n}(t) + \partial_t g^{\mu_1 \cdots \mu_n}(t)\Big) e^{-2t}(-i)^n P_{\mu_1} \dots P_{\mu_n}\end{aligned}$$

what due to (3.12) if $\frac{\partial}{\partial g^{\mu_1 \cdots \mu_n}} \mathcal{S}_t \neq 0$ amounts to

$$\dot{g}^{(n)} = \Big(- ng^{\mu_1 \cdots \mu_n}(t)\Big) e^{-2t}(-i)^n P_{\mu_1} \dots P_{\mu_n} = -ng^{(n)} \ .$$

If $b(t)$ satisfies the RG-equation then δD in the connection D in (3.40) is invariant so that (3.19) in its original from is valid for $g^{(n)}$ as well as $\delta g^{(n)}$. It is interesting to note that the RG-equation displays an $\mathfrak{o}(3,2)$ algebra even if the action parametrized by $\{g^{(n)}\}$ is not conformal. Until now just a linearized flow was considered.

As an example of a z-dependent gauge trafo we take $\epsilon = K$ as in (3.34). Then $\delta B = -K * B + B * K$ or

$$\delta g^{(n)} = -K * g^{(n)} + g^{(n)} * K = -K(z, y) g^{(n)}(z^\alpha, \bar{y}^{\dot{\alpha}}) + g^{(n)}(-z^\alpha, \bar{y}^{\dot{\alpha}}) K(z, y) \ .$$

In this construction $g^{(n)}$ was independent of z^α but this is not a gauge-invariant statement. Since there is no interpretation for z^α in 3D it is suspected [26] that the correct interpretation in 3D is to set $z^\alpha = 0$, thus $\delta g^{(n)} = 0$.

3.5 Adding the inhomogenity

We now return to the right-hand side of (3.1), there the first (tree-level) term contributes a source term of the form (with $g^{\mu}p_{\mu} \equiv g^{\mu_1\cdots\mu_m}p_{\mu_1}\cdots p_{\mu_m}$ and (3.11))

$$
\begin{aligned}
\frac{1}{2}\frac{\delta}{\delta\varphi}\mathcal{S}_t\cdot G\cdot\frac{\delta}{\delta\varphi}\mathcal{S}_t &= \frac{\delta}{\delta\varphi}\int g^{\mu_1\ldots\mu_m}p_{\mu_1}\cdots p_{\mu_m}\varphi(p)\varphi(-p)\cdot\mathcal{K}'(p^2)\cdot\frac{\delta}{\delta\varphi}\int g^{\nu_1\cdots\nu_n}p_{\nu_1}\cdots p_{\nu_n}\varphi(p)\varphi(-p)\\
&= \frac{\delta}{\delta\varphi(p)}\int\frac{d^dq}{(2\pi)^d}g^{\mu_1\ldots\mu_m}q_{\mu_1}\cdots q_{\mu_m}\varphi(q)\varphi(-q)\cdot\mathcal{K}'(p^2)\\
&\quad\cdot\frac{\delta}{\delta\varphi(r)}\int\frac{d^dr}{(2\pi)^d}g^{\nu_1\ldots\nu_n}r_{\nu_1}\cdots r_{\nu_n}\varphi(r)\varphi(-r)\\
&=\int d^dq g^{\mu}q_{\mu}(\delta(p-q)\varphi(-q)+\varphi(q)\delta(p+q))\cdot\mathcal{K}'(p^2)\\
&\quad\cdot\int d^drg^{\nu}r_{\nu}(\delta(p-r)\varphi(-r)+\varphi(r)\delta(p+r))\\
&=\frac{1}{2}2g^{\mu}q_{\mu}\varphi(-p)\cdot\mathcal{K}'(p^2)\cdot 2g^{\nu}q_{\nu}\varphi(-p)\\
&=2\int\frac{d^dp}{(2\pi)^d}\varphi(-p)g^{\mu_1\ldots\mu_m}g^{\nu_1\ldots\nu_n}\mathcal{K}'(p^2)p_{\mu_1}\cdots p_{\mu_m}p_{\nu_1}\cdots p_{\nu_n}\varphi(p)\ . \qquad (3.44)
\end{aligned}
$$

The product of two traceless couplings g needs not to be traceless, thus traceful couplings will be generated along the renormalization group flow. To clarify the algebraic structure, we define "·" on the space of couplings as

$$
g^{\mu_1\ldots\mu_m}\cdot g^{\nu_1\ldots\nu_n}=g^{(\mu_1\ldots\mu_m}g^{\nu_1\ldots\nu_n)}\ , \qquad (3.45)
$$

e.g.

$$
g^{\mu}\cdot g^{\nu}=g^{(\mu}g^{\nu)}=g^{\mu}g^{\nu}+g^{\nu}g^{\mu}
$$

$$
g^{\mu_1\mu_2}\cdot g^{\nu}=g^{(\mu_1\mu_2}g^{\nu)}=g^{\mu_1\mu_2}g^{\nu}+g^{\mu_1\nu}g^{\mu_2}+g^{\nu\mu_2}g^{\mu_1}+g^{\nu\mu_1}g^{\mu_2}+g^{\mu_2\mu_1}g^{\nu}+g^{\mu_2\nu}g^{\mu_1}\ .
$$

Dividing out the ideal generated by elements containing traceful couplings (corresponding to deformations of $\mathcal{S}_t$ that involve the d'Alembertian p^2) then the set of traceless symmetric higher derivative couplings with the above product form an Abelian algebra. If the couplings depend on the coordinates of 2+1 dimensional field theory it is then non-Abelian and upon suitable ordering generate the HS-algebra.

To describe the contribution from (3.44) to the flow of the traceless couplings we expand $\mathcal{K}'$ as $\mathcal{K}'(p^2)=\mathcal{K}'(0)+O(p^2)$. Then, recalling that representation of the momenta p_{μ} in term of spinor variable as in (3.14) takes care of symmetrization and projection onto vanishing trace automatically the non-linear correction to (3.30) can be written as

$$
\partial_t C(t|Y)+[(W_0)_t(t|Y),C(t|Y)]_*=2\mathcal{K}'(0)C(t|Y)C(t|Y)\ . \qquad (3.46)
$$

This is shown in the following. With $g^{(n)}=C(Y|t)$ equation (3.44) gets

$$
\begin{aligned}
\frac{1}{2}\frac{\delta}{\delta\varphi}\mathcal{S}_t\cdot G\cdot\frac{\delta}{\delta\varphi}\mathcal{S}_t&=2\int\frac{d^dp}{(2\pi)^d}\varphi(-p)g^{\mu_1\cdots\mu_m}g^{\nu_1\cdots\nu_n}\mathcal{K}'(0)p_{\mu_1}\cdots p_{\mu_m}p_{\nu_1}\cdots p_{\nu_n}\varphi(p)\\
&=2g^{(n)}(t)\mathcal{K}'(0)g^{(n)}(t)=2\mathcal{K}'(0)C(t|Y)C(t|Y)\ .
\end{aligned}
$$

Equation (3.46) does not look covariant under higher spin gauge transformations (3.36) and (3.37), not even when we restrict ourselves to $\epsilon(x|Y,Z)$ that are independent from Z.

$C(t|Y)C(t|Y)$ can be replaced by $C(t|Y) * C(t|Y)$ because the field C just depends on y^-_α. Thus equation (3.46) is covariant under HS-gauge transformations.

Traceful couplings sourced by (3.44) and sitting in the ideal, do not back react on the traceless couplings at tree-level. We now analyze the situation in presence of interactions. We focus on the quartic term $\lambda \int \varphi(p_1)\cdots\varphi(p_4)\delta^3(p_1+\cdots+p_4)$ in $\mathcal{S}_t$ which is interesting in relation to the O(N) model.

At tree-level, the RG-flow (3.1) produces, among higher order couplings, also traceless couplings of the form

$$2\lambda\mathcal{K}'(0)g^{\mu_1\ldots\mu_n}\int q_{\mu_1}\cdots q_{\mu_n}\varphi(q)\varphi(p_2)\cdots\varphi(p_4)\delta^3(q+\cdots+p_4)\ . \tag{3.47}$$

We show (3.47) by

$$\begin{aligned}
\frac{1}{2}\frac{\delta}{\delta\varphi}\mathcal{S}_t\cdot G\cdot\frac{\delta}{\delta\varphi}\mathcal{S}_t &= \frac{1}{2}\frac{\delta}{\delta\varphi}\Big(\lambda\int\varphi(p_1)\cdots\varphi(p_4)\delta^3(p_1+\cdots+p_4)\Big)\cdot 2\mathcal{K}'(0)\\
&\quad\cdot\frac{\delta}{\delta\varphi}\Big(g^{\mu_1\ldots\mu_n}\int q_{\mu_1}\cdots q_{\mu_n}\varphi(p)\varphi(-p)\Big)\\
&= \frac{1}{2}\lambda\int\Big(((2\pi)^d\delta(q-p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4)+\cdots)\delta^3(p_1+\cdots+p_4)\Big)\cdot 2\mathcal{K}'(0)\\
&\quad\cdot\Big(g^{\mu_1\ldots\mu_n}\int q_{\mu_1}\cdots q_{\mu_n}(2\pi)^d\Big(\delta(q-p)\varphi(-p)+\delta(q+p)\varphi(p)\Big)\\
&= \lambda\int\Big(((2\pi)^d\delta(q-p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4)+\cdots)\delta^3(p_1+\cdots+p_4)\Big)\cdot\mathcal{K}'(0)\\
&\quad\cdot\Big(g^{\mu_1\ldots\mu_n}q_{\mu_1}\cdots q_{\mu_n}2\varphi(-q)\Big)\\
&= 2\lambda\Big(\varphi(p_2)\varphi(p_3)\varphi(p_4)+\cdots\Big)\delta^3(q+\cdots+p_4)\Big)\cdot\mathcal{K}'(0)\cdot\Big(g^{\mu_1\ldots\mu_n}q_{\mu_1}\cdots q_{\mu_n}\varphi(-q)\Big)\\
&= 2\lambda\mathcal{K}'(0)g^{\mu_1\ldots\mu_n}\int q_{\mu_1}\cdots q_{\mu_n}\varphi(q)\varphi(p_2)\cdots\varphi(p_4)\delta^3(q+\cdots+p_4)+\cdots\ .
\end{aligned}$$

These couplings are not captured by the minimal HS-system (3.46) but, again do not effect the running of the traceless, quadratic couplings at tree-level and thus (3.46) is indeed closed at tree-level.

At quantum level (the second term on right-hand side of (3.1)) the above decoupling no longer takes place. At quantum level further investigation is needed, parts of these calculations are shown in [26].

4 Mapping the conformal algebra to de Sitter spacetime

At the end of this chapter we will perform the mapping between the conformal algebra of the Sp($2N$) model and de Sitter spacetime. In subchapter 4.1 we start with the analysis and calculation of the conformal algebra of the O(N) model introduced in 2.3. In the next subchapter 4.2 we map the conformal algebra to Anti-de Sitter and to de Sitter spacetime and see a difference in one of the nine commutators, namely $[K_\mu, P_\nu]$. Therefore we cannot map the O(N) model to de Sitter spacetime. In 4.3 we first try to map a Wick rotated Sp($2N$) model to de Sitter spacetime but this will not lead to the desired flip of the sign in $[K_\mu, P_\nu]$. We will show that behind the obviously simple transformation between the Poincaré coordinates of AdS and dS is hidden something more. We need to link the AdS parametrization (2.3) to the dS parametrization (2.7) and use these transformations to transform the conformal generators in order to map the symmetries of Sp($2N$) to de Sitter spacetime symmetries. Finally we introduce new conformal generators for the higher spin gravity based on the two-component spinors (twistors) respectively the oscillators in order to map them to de Sitter spacetime.

4.1 Conformal algebra of the O(N) model

We now want to calculate the commutation relations of the conformal algebra in 2+1 dimensions ($d = 3$) using the conformal generators of the associated Noether charges of the conformal transformations. For the conformal algebra we need to set $m = 0$. As stated in chapter 2.3, in order to be invariant under scale transformations and for $J_{(K)\nu\mu}$ to be conserved, we either have to use the improved energy-momentum tensor (2.40) or we have to add correction terms to the currents $J_{(D)\mu}$ and $J_{(K)\nu\mu}$. Both approaches are equivalent, whereas we will choose the latter one, as it provides more insight and looks more elegant.

Let us now derive the correction terms for the currents. If we just would take ${J_{(D)}}^\mu = x^\rho {T^\mu}_\rho$ then the derivative is

$$\partial_\mu {J_{(D)}}^\mu = \partial_\mu \Big(x^\rho {T^\mu}_\rho \Big) = \delta^\rho_\mu {T^\mu}_\rho = {T^\mu}_\mu \ . \tag{4.1}$$

This means, that the energy-momentum tensor has to be traceless (${T^\rho}_\rho = 0$) in order that $\partial_\mu {J_{(D)}}^\mu$ vanishes. The other option is to add a correction term J^μ as

$${J_{(D)}}^\mu = x^\rho {T^\mu}_\rho - J^\mu \tag{4.2}$$

so that $\partial_\mu {J_{(D)}}^\mu = 0$. This can be achieved with

$$J^\mu = \frac{2-d}{2} \phi \partial^\mu \phi \left(= -\delta \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} \phi \text{ with } \delta = \frac{d-2}{2} \text{ the canonical dimension} \right) . \tag{4.3}$$

Let us test this for $d = 3$:

$$\partial_\mu {J_{(D)}}^\mu = \partial_\mu \left(x^\rho {T^\mu}_\rho + \frac{1}{2} \phi \partial^\mu \phi \right) = {T^\mu}_\mu + x^\rho \partial_\mu {T^\mu}_\rho + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \phi \partial_\mu \partial^\mu \phi \ . \tag{4.4}$$

Using $\partial_\mu {T^\mu}_\rho = 0$, ${T^\mu}_\mu = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi$ and using the equation of motion $\partial_\mu \partial^\mu \phi = \Box \phi = 0$ we get $\partial_\mu {J_{(D)}}^\mu = 0$. We conclude that the Dilatation operator based on the correct conserved current

($\mu = 0$) in $d = 3$ and with $\pi = \partial_0\phi = -\partial^0\phi$ reads

$$D = \int d^2\vec{x}\left(x^\rho T^0{}_\rho - \frac{1}{2}\phi(x)\pi(x)\right) . \tag{4.5}$$

Same can be done for the generator of the special conformal transformations. However, we have to add two correction terms J^ν and $J_\mu{}^\nu$ of the form

$$J_{(K)}{}^\nu{}_\mu = x^2 T^\nu{}_\mu - 2x_\mu(x^\sigma T^\nu{}_\sigma - J^\nu) - 2J_\mu{}^\nu \tag{4.6}$$

so that $\partial_\nu J_{(K)}{}^\nu{}_\mu = 0$. This can be achieved with

$$J_\mu{}^\nu = \frac{2-d}{2}\phi\partial^\nu\phi \quad \text{and} \quad J_\mu{}^\nu = \frac{2-d}{4}\delta^\nu_\mu\phi^2 . \tag{4.7}$$

The conserved current with the correction terms hence reads

$$J_{(K)}{}^\nu{}_\mu = x^2 T^\nu{}_\mu - 2x_\mu x^\sigma T^\nu{}_\sigma + x_\mu(2-d)\phi\partial^\nu\phi - \frac{2-d}{2}\delta^\nu_\mu\phi^2 . \tag{4.8}$$

Setting $\nu = 0$ we obtain the corrected special conformal generator in $d = 3$ as

$$K_\mu = \int d^2\vec{x}\left(x^2 T^0{}_\mu - 2x_\mu x^\sigma T^0{}_\sigma + x_\mu\phi(x)\pi(x) + \frac{1}{2}\delta^0_\mu\phi(x)^2\right) . \tag{4.9}$$

The two other generators P_μ and $L_{\nu\rho}$ remain unchanged

$$P_\mu = \int d^2\vec{x}\; T^0{}_\mu \qquad L_{\nu\rho} = \int d^2\vec{x}\;(x_\nu T^0{}_\rho - x_\rho T^0{}_\nu) . \tag{4.10}$$

While analyzing the conformal algebra we set the mass to zero and use the O(N) model (2.31) with the Lagrangian $\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi$, $T_{0\mu} = \pi\partial_\mu\phi + \eta_{0\mu}\mathcal{L}$ and $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1)$. As stated in chapter 2.3 the canonical commutation relation is $[\phi(\vec{x},t),\pi(\vec{y},t)] = i\delta(\vec{x}-\vec{y})$. We furthermore use the following identities

$$\left[\frac{\partial}{\partial x^i}\phi(\vec{x},t),\pi(\vec{y},t)\right] = i\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}) \ , \quad \frac{\partial x_\mu}{\partial x^\nu} = \eta_{\mu\nu} \ , \quad \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu \ , \quad \partial^i\phi\partial_i\phi = (\vec{\nabla}\phi)^2 . \tag{4.11}$$

From now on we write (x) instead of $(\vec{x},t)$ and if there is just one integration variable we use $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$ and $\phi \equiv \phi(x)$. We have the spacetime indices $\mu,\nu,\dots \in \{0,1,2\}$ and space indices $i,j,\dots \in \{1,2\}$ with $x^2 = x_\mu x^\mu = x_0x^0 + x_ix^i$ and $\frac{\partial x^i}{\partial x^i} = 2$. Calculating with commutators we need the identity

$$[AB,CD] = A[B,C]D + AC[B,D] + [A,C]DB + C[A,D]B . \tag{4.12}$$

As we have seen in chap. 2.3 the energy-momentum tensor with two time indices equals

$$\begin{aligned} T^0{}_0(x) &= -T_{00}(x) = -\Big(\pi(x)\partial_0\phi(x) - \mathcal{L}(x)\Big) = -\pi(x)^2 + \left(\frac{1}{2}\big(\partial_0\phi(x)\big)^2 - \frac{1}{2}\big(\vec{\nabla}\phi(x)\big)^2\right) \\ &= -\frac{1}{2}\pi(x)^2 - \frac{1}{2}\big(\vec{\nabla}\phi(x)\big)^2 \end{aligned} \tag{4.13}$$

and by integrating it leads to the Hamiltonian $H = -P_0$. Having a time and a space index we get

$$T^0{}_i(x) = -T_{0i}(x) = -\pi(x)\partial_i\phi(x) \tag{4.14}$$

which upon integration leads to the physical momentum P_i.

Before calculating the nine commutator relations of the conformal algebra we calculate the basic building blocks of them, which are

$$\begin{aligned}
[T^0{}_i(x), T^0{}_j(y)] &= [T_{0i}(x), T_{0j}(y)] \\
&= [\pi(x)\partial_i\phi(x), \pi(y)\partial_j\phi(y)] = \pi(x)[\partial_i\phi(x), \pi(y)]\partial_j\phi(y) + \pi(y)[\pi(x), \partial_j\phi(y)]\partial_i\phi(x) \\
&= i\pi(x)\frac{\partial}{\partial y^j}\phi(y)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}) - i\pi(y)\frac{\partial}{\partial x^i}\phi(x)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y}) \ ,
\end{aligned}$$

$$\begin{aligned}
[T^0{}_0(x), T^0{}_j(y)] &= [T_{00}(x), T_{0j}(y)] = \frac{1}{2}\Big[\pi(x)^2 + \big(\vec{\nabla}\phi(x)\big)^2, \pi(y)\partial_j\phi(y)\Big] \\
&= \frac{1}{2}\left(\Big[\pi(x)^2, \pi(y)\partial_j\phi(y)\Big] + \Big[\big(\vec{\nabla}\phi(x)\big)^2, \pi(y)\partial_j\phi(y)\Big]\right) \\
&= \frac{1}{2}\left(2\pi(x)\pi(y)\left[\pi(x), \partial_j\phi(y)\right] + 2\vec{\nabla}\phi(x)\partial_j\phi(y)\left[\vec{\nabla}\phi(x), \pi(y)\right]\right) \\
&= -i\pi(x)\pi(y)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y}) + i\vec{\nabla}_x\phi(x)\frac{\partial}{\partial y^j}\phi(y)\vec{\nabla}_x\delta(\vec{x}-\vec{y}) \ ,
\end{aligned}$$

$$\begin{aligned}
[T^0{}_0(x), T^0{}_0(y)] &= [T_{00}(x), T_{00}(y)] = \frac{1}{4}\Big[\pi(x)^2 + \big(\vec{\nabla}\phi(x)\big)^2, \pi(y)^2 + \big(\vec{\nabla}\phi(y)\big)^2\Big] \\
&= \frac{1}{4}\left(\Big[\pi(x)^2, \big(\vec{\nabla}\phi(y)\big)^2\Big] + \Big[\big(\vec{\nabla}\phi(x)\big)^2, \pi(y)^2\Big]\right) \\
&= \frac{1}{4}\left(4\pi(x)\vec{\nabla}\phi(y)\left[\pi(x), \vec{\nabla}\phi(y)\right] + 4\vec{\nabla}\phi(x)\pi(y)\left[\vec{\nabla}\phi(x), \pi(y)\right]\right) \\
&= -i\pi(x)\vec{\nabla}_y\phi(y)\vec{\nabla}_y\delta(\vec{x}-\vec{y}) + i\vec{\nabla}_x\phi(x)\pi(y)\vec{\nabla}_x\delta(\vec{x}-\vec{y}) \ .
\end{aligned}$$

We see that in these building blocks we can also use T_{00} and T_{0i} with both lower indices as the two minuses drop out in the commutators.

We start with the commutator $[D, P_\mu]$. If μ is a space index, i.e. $\mu = j$, the commutator yields

$$\begin{aligned}
[D, P_j] &= \int d^2\vec{x}d^2\vec{y}\ \left(x^\rho[T_{0\rho}, T_{0j}] - \frac{1}{2}[\phi(x)\pi(x), T^0{}_j]\right) \\
&= \int d^2\vec{x}d^2\vec{y}\Big(x^0[T_{00}, T_{0j}] + x^i[T_{0i}, T_{0j}] + \frac{1}{2}[\phi(x)\pi(x), \pi(y)\partial_j\phi(y)]\Big) \\
&= i\int d^2\vec{x}d^2\vec{y}\bigg(-x^0\pi(x)\pi(y)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y}) + x^0\vec{\nabla}_x\phi(x)\frac{\partial}{\partial y^j}\phi(y)\vec{\nabla}_x\delta(\vec{x}-\vec{y}) \\
&\qquad + x^i\pi(x)\frac{\partial}{\partial y^j}\phi(y)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}) - x^i\pi(y)\frac{\partial}{\partial x^i}\phi(x)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y})\bigg) \\
&\quad + i\int d^2\vec{x}\left(\frac{1}{2}\pi(x)\partial_j\phi(x) + \frac{1}{2}\phi(x)\partial_j\pi(x)\right) \ .
\end{aligned}$$

The last two term cancel each other out and partial integration leads to

$$[D,P_j] = i\int d^2\vec{x}d^2\vec{y}\bigg(x^0\pi(x)\frac{\partial}{\partial y^j}\pi(y)\delta(\vec{x}-\vec{y}) - x^0\vec{\nabla}_x\vec{\nabla}_x\phi(x)\frac{\partial}{\partial y^j}\phi(y)\delta(\vec{x}-\vec{y})$$

$$- \frac{\partial x^i}{\partial x^i}\pi(x)\frac{\partial}{\partial y^j}\phi(y)\delta(\vec{x}-\vec{y}) - x^i\frac{\partial}{\partial x^i}\pi(x)\frac{\partial}{\partial y^j}\phi(y)\delta(\vec{x}-\vec{y}) + x^i\frac{\partial}{\partial y^j}\pi(y)\frac{\partial}{\partial x^i}\phi(x)\delta(\vec{x}-\vec{y})\bigg)$$

$$= i\int d^2\vec{x}\Big(x^0\pi\partial_j\pi - x^0\vec{\nabla}\vec{\nabla}\phi\partial_j\phi - 2\pi\partial_j\phi - x^i\partial_i\pi\partial_j\phi + x^i\partial_j\pi\partial_i\phi\Big)\ .$$

With yet another partial integration on the last two terms we finally get

$$[D,P_j] = i\int d^2\vec{x}\Big(x^0\pi\partial_j\pi - x^0\vec{\nabla}\vec{\nabla}\phi\partial_j\phi - 2\pi\partial_j\phi + 2\pi\partial_j\phi + x^i\pi\partial_i\partial_j\phi - \delta^i_j\pi\partial_i\phi - x^i\pi\partial_j\partial_i\phi\Big)$$

$$= i\int d^2\vec{x}\Big(\frac{1}{2}x^0\partial_j(\pi^2) + x^0\vec{\nabla}\phi\partial_j\vec{\nabla}\phi - \pi\partial_j\pi\Big) = i\int d^2\vec{x}\Big(\frac{1}{2}x^0\partial_j(\pi^2) + \frac{1}{2}x^0\partial_j(\vec{\nabla}\phi)^2 - \pi\partial_j\pi\Big)$$

$$= i\int d^2\vec{x}\Big(-\pi\partial_j\phi\Big) = i\int d^2\vec{x}T^0{}_j = iP_j\ .$$

For $\mu = 0$ we calculate

$$[D,P_0] = \int d^2\vec{x}d^2\vec{y}\ \Big(x^\rho[T_{0\rho},T_{00}] - \frac{1}{2}[\phi(x)\pi(x),T^0{}_0]\Big)$$

$$= \int d^2\vec{x}d^2\vec{y}\ \Big(x^0[T_{00},T_{00}] + x^i[T_{0i},T_{00}] + \frac{1}{4}[\phi(x)\pi(x),\pi(y)^2] + \frac{1}{4}[\phi(x)\pi(x),(\vec{\nabla}\phi(y))^2]\Big)$$

$$= i\int d^2\vec{x}d^2\vec{y}\bigg(-x^0\pi(x)\vec{\nabla}_y\phi(y)\vec{\nabla}_y\delta(\vec{x}-\vec{y}) + x^0\vec{\nabla}_x\phi(x)\pi(y)\vec{\nabla}_x\delta(\vec{x}-\vec{y})$$

$$+ x^i\pi(x)\pi(y)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}) - x^i\vec{\nabla}_y\phi(y)\frac{\partial}{\partial x^i}\phi(x)\vec{\nabla}_y\delta(\vec{x}-\vec{y})\bigg)$$

$$+ i\int d^2\vec{x}\left(\frac{1}{2}\pi(x)^2 + \frac{1}{2}\phi(x)\vec{\nabla}\vec{\nabla}\phi(x)\right)\ .$$

Partial integration in the first integral and in the last term of the second integral leads to

$$[D,P_0] = i\int d^2\vec{x}\left(x^0\pi\vec{\nabla}\vec{\nabla}\phi - x^0\vec{\nabla}\vec{\nabla}\phi\pi - 2\pi^2 - x^i\partial_i\pi\pi + x^i\vec{\nabla}\vec{\nabla}\phi\partial_i\phi + \frac{1}{2}\pi^2 - \frac{1}{2}(\vec{\nabla}\phi)^2\right)$$

$$= i\int d^2\vec{x}\left(-2\pi^2 - x^i\partial_i\pi\pi + x^i\vec{\nabla}\vec{\nabla}\phi\partial_i\phi + \frac{1}{2}\pi^2 - \frac{1}{2}(\vec{\nabla}\phi)^2\right)$$

$$= i\int d^2\vec{x}\left(-2\pi^2 - x^i\partial_i\pi\pi - x^i\vec{\nabla}\phi\partial_i\vec{\nabla}\phi - \vec{\nabla}x^i\vec{\nabla}\phi\partial_i\phi + \frac{1}{2}\pi^2 - \frac{1}{2}(\vec{\nabla}\phi)^2\right)$$

$$= i\int d^2\vec{x}\left(-2\pi^2 - \frac{x^i}{2}\partial_i(\pi^2) - \frac{x^i}{2}\partial_i((\vec{\nabla}\phi)^2) - \partial^i\phi\partial_i\phi + \frac{1}{2}\pi^2 - \frac{1}{2}(\vec{\nabla}\phi)^2\right)$$

$$= i\int d^2\vec{x}\left(-2\pi^2 + \pi^2 + (\vec{\nabla}\phi)^2 - (\vec{\nabla}\phi)^2 + \frac{1}{2}\pi^2 - \frac{1}{2}(\vec{\nabla}\phi)^2\right)$$

$$= i\int d^2\vec{x}\left(-\frac{1}{2}\pi^2 - \frac{1}{2}(\vec{\nabla}\phi)^2\right) = i\int d^2\vec{x}\ T^0{}_0 = iP_0\ .$$

Combining the results for $\mu = j$ and $\mu = 0$ we obtain

$$[D,P_\mu] = iP_\mu\ . \tag{4.15}$$

We continue with the commutator $[K_\mu, P_\nu]$. Here we have four different cases. First we set $\mu = i$ and $\nu = j$ and receive

$$\begin{aligned}
[K_i, P_j] &= \int d^2\vec{x} d^2\vec{y} \Big(x^2 [T_{0i}, T_{0j}] - 2x_i x^\sigma [T_{0\sigma}, T_{0j}] - x_i [\phi(x)\pi(x), \pi(y)\partial_j\phi(y)] - \frac{1}{2}\delta_i^0 [\phi(x)^2, \pi(y)\partial_j\phi(y)] \Big) \\
&= \int d^2\vec{x} d^2\vec{y} \Big(x^2 [T_{0i}, T_{0j}] - 2x_i x^0 [T_{00}, T_{0j}] - 2x_i x^k [T_{0k}, T_{0j}] \\
&\qquad - x_i [\phi(x), \pi(y)] \pi(x) \partial_j \phi(y) - x_i [\pi(x), \partial_j\phi(y)] \phi(x)\pi(y) - \delta_i^0 [\phi(x), \pi(y)] \phi(x) \partial_j \phi(y) \Big) \\
&= i \int d^2\vec{x} d^2\vec{y} \Big(x^2 \pi(x) \frac{\partial}{\partial x^i} \delta(\vec{x} - \vec{y}) \frac{\partial}{\partial y^j} \phi(y) - x^2 \pi(y) \frac{\partial}{\partial y^j} \delta(\vec{x} - \vec{y}) \frac{\partial}{\partial x^i} \phi(x) \\
&\quad + 2x_i x^0 \pi(x)\pi(y) \frac{\partial}{\partial y^j} \delta(\vec{x} - \vec{y}) - 2x_i x^0 \vec{\nabla}_x \phi(x) \frac{\partial}{\partial y^j} \phi(y) \vec{\nabla}_x \delta(\vec{x} - \vec{y}) \\
&\quad - 2x_i x^k \pi(x) \frac{\partial}{\partial x^k} \delta(\vec{x} - \vec{y}) \frac{\partial}{\partial y^j} \phi(y) + 2x_i x^k \pi(y) \frac{\partial}{\partial y^j} \delta(\vec{x} - \vec{y}) \frac{\partial}{\partial x^k} \phi(x) \Big) \\
&\quad + i \int d^2\vec{x} \big(- x_i \pi \partial_j \phi - x_i \phi \partial_j \pi - \delta_i^0 \phi \partial_j \phi \big) \\
&= i \int d^2\vec{x} \Big(- 2x_i \pi \partial_j \phi - x^2 \partial_i \pi \partial_j \phi + x^2 \partial_j \pi \partial_i \phi \\
&\quad - 2x_i x^0 \pi \partial_j \pi + 2\vec{\nabla} x_i x^0 \vec{\nabla} \phi \partial_j \phi + 2x_i x^0 \vec{\nabla}\vec{\nabla}\phi \partial_j \phi \\
&\quad + 2x_i \pi \partial_j \phi + 4x_i \pi \partial_j \phi + 2x_i x^k \partial_k \pi \partial_j \phi - 2x_i x^k \partial_j \pi \partial_k \phi - \delta_i^0 \phi \delta_j^0 \partial_0 \phi \Big) \\
&= i \int d^2\vec{x} \Big(2x_i \pi \partial_j \phi - 2x_j \pi \partial_i \phi + \eta_{ij} x^0 \pi^2 + 2x^0 \partial_i \phi \partial_j \phi + \eta_{ij} x^0 (\vec{\nabla}\phi)^2 - 2\vec{\nabla} x_i x^0 (\vec{\nabla}\phi) \partial_j \phi \\
&\quad + 4x_i \pi \partial_j \phi - 2x_i \pi \partial_j \phi - 4x_i \pi \partial_j \phi + 2\eta_{ij} x^k \pi \partial_k \phi + 2x_i \pi \partial_j \phi + \eta_{ij} \phi \pi \Big) \\
&= i \int d^2\vec{x} \Big(\eta_{ij} x^0 \Big(\pi^2 + (\vec{\nabla}\phi)^2 \Big) + \eta_{ij} \phi\pi + 2\eta_{ij} x^k \pi \partial_k \phi + 2x_i \pi \partial_j \phi - 2x_j \pi \partial_i \phi \Big) \\
&= -2i\eta_{ij} D - 2i L_{ij} \ .
\end{aligned}$$

For $\mu = i$ and $\nu = 0$ we get

$$\begin{aligned}
[K_i, P_0] &= \int d^2\vec{x}d^2\vec{y}\Big(x^2[T_{0i}, T_{00}] - 2x_i x^\sigma[T_{0\sigma}, T_{00}] - \frac{x_i}{2}[\phi(x)\pi(x), \pi(y)^2] - \frac{x_i}{2}[\phi(x)\pi(x), (\vec{\nabla}\phi(y))^2] \\
&\quad - \frac{1}{4}\delta_i^0[\phi(x)^2, \pi(y)^2] - \frac{1}{4}\delta_i^0[\phi(x)^2, (\vec{\nabla}\phi(y))^2]\Big) \\
&= \int d^2\vec{x}d^2\vec{y}\Big(x^2[T_{0i}, T_{00}] - 2x_i x^0[T_{00}, T_{00}] - 2x_i x^k[T_{0k}, T_{00}]\Big) \\
&\quad + i\int d^2\vec{x}\Big(- x_i\pi^2 - x_i(\vec{\nabla}\phi)^2 - \delta_i^0\phi\pi\Big) \\
&= i\int d^2\vec{x}d^2\vec{y}\Big(x^2\pi(x)\pi(y)\frac{\partial}{\partial x^i}\delta(x-y) - x^2\vec{\nabla}_y\phi(y)\frac{\partial}{\partial x^i}\phi(x)\vec{\nabla}_y\delta(x-y) \\
&\quad + 2x_i x^0\pi(x)\vec{\nabla}_y\phi(y)\vec{\nabla}_y\delta(x-y) - 2x_i x^0\vec{\nabla}_x\phi(x)\pi(y)\vec{\nabla}_x\delta(x-y) \\
&\quad - 2x_i x^k\pi(x)\pi(y)\frac{\partial}{\partial x^k}\delta(x-y) + 2x_i x^k\vec{\nabla}_y\phi(y)\frac{\partial}{\partial x^k}\phi(x)\vec{\nabla}_y\delta(x-y)\Big) \\
&\quad + i\int d^2\vec{x}\Big(- x_i\pi^2 - x_i(\vec{\nabla}\phi)^2 - \delta_i^0\phi\partial_0\phi\Big) \\
&= i\int d^2\vec{x}\Big(- 2x_i\pi^2 - x^2\partial_i\pi\pi + x^2\vec{\nabla}\vec{\nabla}\phi\partial_i\phi - 2x_i x^0\pi\vec{\nabla}\vec{\nabla}\phi + 2\vec{\nabla}x_i x^0\vec{\nabla}\phi\pi + 2x_i x^0\vec{\nabla}\vec{\nabla}\phi\pi \\
&\quad + 2x_i\pi^2 + 4x_i\pi^2 + 2x_i x^k\partial_k\pi\pi - 2x_i x^k\vec{\nabla}\vec{\nabla}\phi\partial_k\phi - x_i\pi^2 - x_i(\vec{\nabla}\phi)^2 - \frac{1}{2}\partial_i(\phi^2)\Big) \\
&= i\int d^2\vec{x}\Big(- \frac{1}{2}x^2\partial_i(\pi^2) - \frac{1}{2}x^2\partial_i((\vec{\nabla}\phi)^2) - 2\vec{\nabla}x_k x^k\vec{\nabla}\phi\partial_i\phi + 2x^0\partial_i\phi\pi + 4x_i\pi^2 \\
&\quad + x_i x^k\partial_k(\pi^2) + x_i x^k\partial_k(\vec{\nabla}\phi)^2 + 2\vec{\nabla}x_i x^k\vec{\nabla}\phi\partial_k\phi + 2x_i\vec{\nabla}x^k\vec{\nabla}\phi\partial_k\phi - x_i\pi^2 - x_i(\vec{\nabla}\phi)^2\Big) \\
&= i\int d^2\vec{x}\Big(x_i\pi^2 + x_i(\vec{\nabla}\phi)^2 - 2x_0\partial_i\phi\pi\Big) = -2i\int d^2\vec{x}\Big(x_i T^0{}_0 - x_0 T_i^0\Big) \\
&= 2iL_{i0} = -2i\eta_{i0}D - 2iL_{i0}\ .
\end{aligned}$$

For $\mu = 0$ and $\nu = j$ we get

$$
\begin{aligned}
[K_0, P_j] =& \int d^2\vec{x} d^2\vec{y} \Big(x^2 [T_{00}, T_{0j}] - 2x_0 x^\sigma [T_{0\sigma}, T_{0j}] + x_0 [\phi(x)\pi(x), \pi(y)\partial_j\phi(y)] + \frac{1}{2}\delta_0^0 [\phi(x)^2, \pi(y)\partial_j\phi(y)] \Big) \\
=& \int d^2\vec{x} d^2\vec{y} \Big(x^2 [T_{00}, T_{0j}] - 2x_0 x^0 [T_{00}, T_{0j}] - 2x_0 x^k [T_{0k}, T_{0j}] \Big) \\
& + i \int d^2\vec{x} \Big(x_0 \pi \partial_j \phi + x_0 \phi \partial_j \pi + \frac{1}{2} \partial_j (\phi^2) \Big) \\
=& \, i \int d^2\vec{x} d^2\vec{y} \Big(- x^2 \pi(x)\pi(y) \frac{\partial}{\partial y^j} \delta(x-y) + x^2 \vec{\nabla}_x \phi(x) \frac{\partial}{\partial y^j} \phi(y) \vec{\nabla}_x \delta(x-y) \\
& + x_0 x^0 \pi(x)\pi(y) \frac{\partial}{\partial y^j} \delta(x-y) - 2x_0 x^0 \vec{\nabla}_x \phi(x) \frac{\partial}{\partial y^j} \phi(y) \vec{\nabla}_x \delta(x-y) \\
& - 2x_0 x^k \pi(x) \frac{\partial}{\partial y^j} \phi(y) \frac{\partial}{\partial x^k} \delta(\vec{x}-\vec{y}) + 2x_0 x^k \pi(y) \frac{\partial}{\partial x^k} \phi(x) \frac{\partial}{\partial y^j} \delta(\vec{x}-\vec{y}) \Big) \\
=& \, i \int d^2\vec{x} \Big(x^2 \pi \partial_j \pi - 2\vec{\nabla} x_k x^k \vec{\nabla}\phi \partial_j \phi - x^2 \vec{\nabla}\vec{\nabla}\phi \partial_j \phi - 2x_0 x^0 \pi \partial_j \pi + 2x_0 x^0 \vec{\nabla}\vec{\nabla}\phi \partial_j \phi \\
& + 2x_0 \pi \partial_j \phi + 2x_0 x^k \partial_k \pi \partial_j \phi - 2x_0 x^k \partial_j \pi \partial_k \phi \Big) \\
=& \, i \int d^2\vec{x} \Big(2x_0 \pi \partial_j \phi - x_j \pi^2 - x_j (\vec{\nabla}\phi)^2 \Big) = 2i \int d^2\vec{x} \Big(x_0 T^0{}_j - x_j T^0{}_0 \Big) \\
=& -2iL_{0j} = -2i\eta_{0j} D - 2iL_{0j} \; .
\end{aligned}
$$

Finally for $\mu = 0$ and $\nu = 0$ we calculate

$$
\begin{aligned}
[K_0, P_0] &= \int d^2\vec{x} d^2\vec{y} \Big(x^2[T_{00}, T_{00}] - 2x_0 x^\sigma [T_{0\sigma}, T_{00}] - \frac{x_0}{2}[\phi(x)\pi(x), \pi(y)^2] - \frac{x_0}{2}[\phi(x)\pi(x), (\vec{\nabla}\phi(y))^2] \\
&\quad - \frac{1}{4}\delta^0_0[\phi(x)^2, \pi(y)^2] - \frac{1}{4}\delta^0_0[\phi(x)^2, (\vec{\nabla}\phi(y))^2 \Big) \\
&= \int d^2\vec{x} d^2\vec{y} \Big(x^2[T_{00}, T_{00}] - 2x_0 x^0 [T_{00}, T_{00}] - 2x_0 x^k [T_{0k}, T_{00}]] \Big) \\
&\quad + i \int d^2\vec{x} \Big(- x_0 \pi^2 - x_0 (\vec{\nabla}\phi)^2 - \phi\pi \Big) \\
&= i \int d^2\vec{x} d^2\vec{y} \Big(- x^2 \pi(x) \vec{\nabla}_y \phi(y) \vec{\nabla}_y \delta(x-y) + x^2 \vec{\nabla}_x \phi(x) \pi(y) \vec{\nabla}_x \delta(x-y) \\
&\quad + 2x_0 x^0 \pi(x) \vec{\nabla}_y \phi(y) \vec{\nabla}_y \delta(x-y) - 2x_0 x^0 \vec{\nabla}_x \phi(x) \pi(y) \vec{\nabla}_x \delta(x-y) \\
&\quad - 2x_0 x^k \pi(x)\pi(y) \frac{\partial}{\partial x^k} \delta(x-y) + 2x_0 x^k \vec{\nabla}_y \phi(y) \frac{\partial}{\partial x^k} \phi(x) \vec{\nabla}_y \delta(x-y) \Big) \\
&\quad + i \int d^2\vec{x} \Big(- x_0 \pi^2 - x_0 (\vec{\nabla}\phi)^2 - \phi\pi \Big) \\
&= i \int d^2\vec{x} \Big(x^2 \pi \vec{\nabla}\vec{\nabla}\phi - 2\vec{\nabla} x_k x^k \vec{\nabla}\phi\pi - x^2 \vec{\nabla}\vec{\nabla}\phi\pi - 2x_0 x^0 \pi \vec{\nabla}\vec{\nabla}\phi + 2x_0 x^0 \vec{\nabla}\vec{\nabla}\phi\pi \\
&\quad + 4x_0 \pi^2 + 2x_0 x^k \partial_k \pi\pi - 2x_0 x^k \vec{\nabla}\vec{\nabla}\phi \partial_k \phi - x_0 \pi^2 - x_0 (\vec{\nabla}\phi)^2 - \phi\pi \Big) \\
&= i \int d^2\vec{x} \Big(- 2x^k \partial_k \phi\pi + 4x_0 \pi^2 - 2x_0 \pi^2 + 2x_0 \vec{\nabla} x^k \vec{\nabla}\phi \partial_k \phi + 2x_0 x^k \vec{\nabla}\phi \partial_k \vec{\nabla}\phi - x_0 \pi^2 - x_0 (\vec{\nabla}\phi)^2 - \phi\pi \Big) \\
&= i \int d^2\vec{x} \Big(- x^0 \pi^2 - x^0 (\vec{\nabla}\phi)^2 - 2x^k \pi \partial_k \phi - \phi\pi \Big) = 2i \int d^2\vec{x} \Big(x^0 T^0{}_0 + x^k T^0{}_k - \phi\pi \Big) \\
&= 2iD = -2i\eta_{00} D - 2iL_{00} \ .
\end{aligned}
$$

Combining these four result we get

$$
[K_\mu, P_\nu] = -2i\eta_{\mu\nu} D - 2iL_{\mu\nu} \ . \tag{4.16}
$$

The calculation of the other seven commutators are shown in appendix C.3. The complete conformal algebra of the linear O(N) model reads

$$
\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= -2i\eta_{\mu\nu} D - 2iL_{\mu\nu} \\
[L_{\mu\nu}, L_{\rho\sigma}] &= -i\Big(\eta_{\mu\sigma} L_{\nu\rho} + \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} - \eta_{\mu\rho} L_{\nu\sigma} \Big) \\
[L_{\nu\rho}, K_\mu] &= -i\Big(\eta_{\rho\mu} K_\nu - \eta_{\nu\mu} K_\rho \Big) \\
[L_{\nu\rho}, P_\mu] &= -i\Big(\eta_{\rho\mu} P_\nu - \eta_{\nu\mu} P_\rho \Big) \\
[D, L_{\mu\nu}] &= 0 \\
[P_\mu, P_\rho] &= 0 \\
[K_\mu, K_\rho] &= 0 \ .
\end{aligned} \tag{4.17}
$$

We see that this conformal algebra is identical to the higher spin conformal algebra (3.17) built from the two-component spinors (twistors) except for a factor $-i$.

4.2 Mapping the conformal algebra to AdS and dS spacetime

We now want to map the 3 dimensional conformal algebra (4.17) of the CFT$_3$ O(N) model to the algebra $\mathfrak{so}(3,2)$ of the AdS$_4$ group SO$(3,2)$.

The generators $L_{\mu\nu}$ form the Lorentz algebra $\mathfrak{so}(2,1)$ as a subalgebra. The generators of the conformal algebra can be grouped in a way that it equates the algebra $\mathfrak{so}(3,2)$ [2]. The generators of $\mathfrak{so}(3,2)$ are denoted by $L_{MN} = -L_{NM}$, where M and N run from 0 to 4 and have to satisfy the algebra

$$[L_{MN}, L_{OP}] = -i\Big(\eta_{MP}J_{NO} + \eta_{NO}J_{MP} - \eta_{MO}J_{NP} - \eta_{NP}J_{MO}\Big) \tag{4.18}$$

where $\eta = \text{diag}(-1,1,1)$ is replaced by $\bar{\eta} = \text{diag}(-1,1,1,-1,1)$. The generators $L_{\mu\nu}$ with $\mu,\nu \in \{0,1,2\}$ are the generators of the Lorentz group and satisfy the usual commutation relation with $\eta = \text{diag}(-1,1,1)$

$$[L_{\mu\nu}, L_{\rho\sigma}] = -i\Big(\eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\rho}L_{\mu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma}\Big)\ . \tag{4.19}$$

In order to map the conformal algebra (4.17) to the algebra $\mathfrak{so}(3,2)$ we have to construct the map between the remaining $\mathfrak{so}(3,2)$ generators $L_{\mu 3}$, $L_{\mu 4}$ and L_{34} and the generators D, P_μ and K_μ of the conformal algebra. As L_{34} transforms as a scalar under the Lorentz transformations $\mathfrak{so}(2,1)$, it has to commute with $L_{\mu\nu}$ and therefore we can identify

$$L_{34} = D\ . \tag{4.20}$$

Due to the fact that $L_{\mu 3}$ and $L_{\mu 4}$ are vectors under Lorentz transformations $\mathfrak{so}(2,1)$ we can relate them to P_μ and K_μ using the following identifications

$$L_{\mu 3} = \frac{1}{2}(K_\mu + P_\mu) \qquad , \qquad L_{\mu 4} = \frac{1}{2}(K_\mu - P_\mu) \tag{4.21}$$

which correspond to

$$K_\mu = L_{\mu 3} + L_{\mu 4} \qquad , \qquad P_\mu = L_{\mu 3} - L_{\mu 4}\ . \tag{4.22}$$

Let us now calculate the commutators of the conformal algebra in AdS spacetime with $\bar{\eta} \equiv \eta = \text{diag}(-1,1,1,-1,1)$, thus $\eta_{33} = -1$ and $\mu,\nu \in \{0,1,2\}$. We obtain

$$\begin{aligned}
[D, P_\mu] &= [L_{34}, L_{\mu 3}] - [L_{34}, L_{\mu 4}] \\
&= -i\Big(\eta_{33}L_{4\mu} + \eta_{4\mu}L_{33} - \eta_{3\mu}L_{43} - \eta_{43}L_{3\mu}\Big) + i\Big(\eta_{34}L_{4\mu} + \eta_{4\mu}L_{34} - \eta_{3\mu}L_{44} - \eta_{44}L_{4\mu}\Big) \\
&= -i\Big(\eta_{33}L_{4\mu} + \eta_{44}L_{3\mu}\Big) = -i\Big(L_{\mu 4} - L_{\mu 3}\Big) = iP_\mu\ ,
\end{aligned}$$

$$\begin{aligned}
[D, K_\mu] &= [L_{34}, L_{\mu 3}] + [L_{34}, L_{\mu 4}] \\
&= -i\Big(\eta_{33}L_{4\mu} + \eta_{4\mu}L_{33} - \eta_{3\mu}L_{43} - \eta_{43}L_{3\mu}\Big) - i\Big(\eta_{34}L_{4\mu} + \eta_{4\mu}L_{34} - \eta_{3\mu}L_{44} - \eta_{44}L_{3\mu}\Big) \\
&= -i\Big(\eta_{33}L_{4\mu} - \eta_{44}L_{3\mu}\Big) = -i\Big(L_{\mu 4} + L_{\mu 3}\Big) = -iK_\mu\ ,
\end{aligned}$$

$$
\begin{aligned}
[K_\mu, P_\nu] &= [L_{\mu 3} + L_{\mu 4}, L_{\nu 3} - L_{\nu 4}] \\
&= -i\Big(\eta_{\mu 3} L_{3\nu} + \eta_{3\nu} L_{\mu 3} - \eta_{\mu\nu} L_{33} - \eta_{33} L_{\mu\nu} - \eta_{\mu 4} L_{3\nu} - \eta_{3\nu} L_{\mu 4} + \eta_{\mu\nu} L_{34} + \eta_{34} L_{\mu\nu} \\
&\quad + \eta_{\mu 3} L_{4\nu} + \eta_{4\nu} L_{\mu 3} - \eta_{\mu\nu} L_{43} - \eta_{43} L_{\mu\nu} - \eta_{\mu 4} L_{4\nu} - \eta_{4\nu} L_{\mu 4} + \eta_{\mu\nu} L_{44} + \eta_{44} L_{\mu\nu}\Big) \\
&= -i\Big(- \eta_{33} L_{\mu\nu} + \eta_{\mu\nu} L_{34} - \eta_{\mu\nu} L_{43} + \eta_{44} L_{\mu\nu}\Big) = 2i\eta_{\mu\nu} L_{43} - 2iL_{\mu\nu} \\
&= -2i\eta_{\mu\nu} D - 2iL_{\mu\nu} \ .
\end{aligned}
$$

We see that the commutators are the same as in the conformal algebra (4.17), the calculation of the remaining six are not shown. This statement fits to the conjectured Giombi-Klebanov-Polyakov-Yin duality relating the $O(N)$ CFT_3 to the Vasiliev higher spin gravity in AdS_4 (see [8] and chapter 1).

Now we want to map the CFT_3 conformal algebra to the algebra $\mathfrak{so}(4,1)$ of the dS_4 group $SO(4,1)$. In the algebra (4.18) we now use $\bar{\eta} = \mathrm{diag}(-1,1,1,1,1)$ and we make the identifications

$$
L_{34} = -iD \quad , \quad L_{\mu 3} = -\frac{i}{2}(K_\mu - P_\mu) \quad , \quad L_{\mu 4} = \frac{1}{2}(K_\mu + P_\mu) \ . \tag{4.23}
$$

This corresponds to

$$
D = iL_{34} \quad , \quad K_\mu = L_{\mu 4} + iL_{\mu 3} \quad , \quad P_\mu = L_{\mu 4} - iL_{\mu 3} \ . \tag{4.24}
$$

Let us now calculate the commutators of the conformal algebra in dS spacetime with $\bar{\eta} \equiv \eta = \mathrm{diag}(-1,1,1,1,1)$, thus $\eta_{33} = 1$ and $\mu, \nu \in \{0,1,2\}$. We obtain

$$
\begin{aligned}
[D, P_\mu] &= [iL_{34}, L_{\mu 4}] - [iL_{34}, iL_{\mu 3}] = i[L_{34}, L_{\mu 4}] + [L_{34}, L_{\mu 3}] \\
&= \Big(\eta_{34} L_{4\mu} + \eta_{4\mu} L_{34} - \eta_{3\mu} L_{44} - \eta_{44} L_{3\mu}\Big) - i\Big(\eta_{33} L_{4\mu} + \eta_{4\mu} L_{33} - \eta_{3\mu} L_{43} - \eta_{43} L_{3\mu}\Big) \\
&= -\eta_{44} L_{3\mu} - i\eta_{33} L_{4\mu} = i(L_{\mu 4} - iL_{\mu 3}) = iP_\mu
\end{aligned}
$$

$$
\begin{aligned}
[D, K_\mu] &= [iL_{34}, L_{\mu 4}] + [iL_{34}, iL_{\mu 3}] = i[L_{34}, L_{\mu 4}] - [L_{34}, L_{\mu 3}] \\
&= \Big(\eta_{34} L_{4\mu} + \eta_{4\mu} L_{34} - \eta_{3\mu} L_{44} - \eta_{44} L_{3\mu}\Big) + i\Big(\eta_{33} L_{4\mu} + \eta_{4\mu} L_{33} - \eta_{3\mu} L_{43} - \eta_{43} L_{3\mu}\Big) \\
&= -\eta_{44} L_{3\mu} + i\eta_{33} L_{4\mu} = -i(L_{\mu 4} + iL_{\mu 3}) = -iK_\mu
\end{aligned}
$$

$$
\begin{aligned}
[K_\mu, P_\nu] &= [L_{\mu 4} + iL_{\mu 3}, L_{\nu 4} - iL_{\nu 3}] \\
&= -i\Big(\eta_{\mu 4} L_{4\nu} + \eta_{4\nu} L_{\mu 4} - \eta_{\mu\nu} L_{44} - \eta_{44} L_{\mu\nu} - i\eta_{\mu 3} L_{4\nu} - i\eta_{4\nu} L_{\mu 3} + i\eta_{\mu\nu} L_{43} + i\eta_{43} L_{\mu\nu} \\
&\quad + i\eta_{\mu 4} L_{3\nu} + i\eta_{3\nu} L_{\mu 4} - i\eta_{\mu\nu} L_{34} - i\eta_{34} L_{\mu\nu} + \eta_{\mu 3} L_{3\nu} + \eta_{3\nu} L_{\mu 3} - \eta_{\mu\nu} L_{33} - \eta_{33} L_{\mu\nu}\Big) \\
&= -i\Big(- \eta_{44} L_{\mu\nu} + i\eta_{\mu\nu} L_{43} - i\eta_{\mu\nu} L_{34} - \eta_{33} L_{\mu\nu}\Big) = -2\eta_{\mu\nu} L_{34} + 2iL_{\mu\nu} \\
&= 2i\eta_{\mu\nu} D + 2iL_{\mu\nu}
\end{aligned} \tag{4.25}
$$

The calculation of the remaining six commutators is not shown here, but it has the same results as in the conformal algebra (4.17). Thus we see that all commutators of the algebra $\mathfrak{so}(3,1)$ of the dS_4 group are the same as in the $O(N)$ model, except $[K_\mu, P_\nu]$ as seen in (4.25). The commutator $[K_\mu, P_\nu]$ has flipped its sign and therefore does not fit to the conformal algebra of the $O(N)$ model. This fact will proof important in the next subchapter.

4.3 Mapping the conformal algebra of the Sp($2N$) model to de Sitter spacetime

We have seen in chapter 4.2 that the algebra of the de Sitter group SO(3, 1) reads

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i\eta_{\mu\nu}D + 2iL_{\mu\nu} \\
[L_{\mu\nu}, L_{\rho\sigma}] &= -i\Big(\eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\rho}L_{\mu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma}\Big) \\
[L_{\nu\rho}, K_\mu] &= -i\Big(\eta_{\rho\mu}K_\nu - \eta_{\nu\mu}K_\rho\Big) \\
[L_{\nu\rho}, P_\mu] &= -i\Big(\eta_{\rho\mu}P_\nu - \eta_{\nu\mu}P_\rho\Big) \\
[D, L_{\mu\nu}] &= 0 \\
[P_\mu, P_\rho] &= 0 \\
[K_\mu, K_\rho] &= 0 \ .
\end{aligned} \tag{4.26}$$

It only differs for $[K_\mu, P_\nu]$ from the O(N) conformal algebra (4.17), as it has a flipped sign for this commutator.

Dynamics of a dS/CFT correspondence

Before we search for a mapping of symmetries between the Sp($2N$) model and de Sitter spacetime we will take a look at the dynamics of a possible dS/CFT correspondence. As our universe is unlikely to have an AdS boundary, it may well have a dS boundary in the far future and therefore it is very valid to look for a dS/CFT duality. One important difference is that in AdS/CFT the radial direction z emerges holographically from the CFT, while in dS/CFT time itself must be holographically emergent. It is conjectured that Vasiliev's higher spin gravity in dS_4 is holographically dual to the non-unitary 3-dimensional conformal field theory (CFT_3) of the Sp($2N$) model with anticommuting scalars presented in chapter 2.3 [8]. This dS/CFT duality is the analogue to the conjectured O(N)-higher spin duality of Giombi-Klebanov-Polyakov-Yin [14, 15].

The appearance of the Sp($2N$) model can be understood as an analytic continuation of the O(N) model under $N \to -N$. The singlet correlators of the Sp($2N$) model are obtained by replacing N with $-N$ in the expressions for the corresponding O(N) correlators. This relation holds due to the fact that closed loops which give powers of N are always accompanied by an extra minus sign from the fermionic statistics in the Sp($2N$) theory [8, 13]. The reason for the extra minus starts with the fact that for a fermionic field the functional integral gives a $\det A$ in the integration measure and not $\frac{1}{\det A}$ like for a bosonic field. Related to free energy and finite size effects which are a probe of the unitarity of a theory, we consider the action S_χ in (2.43) in Euclidean space as

$$S_\chi = \int d^3x \partial\bar{\chi}\partial\chi \ , \tag{4.27}$$

where χ is an N-component massless complex fermion field without interaction. Now we can perform a functional integral to obtain the partition function

$$\mathcal{Z} = \int \mathcal{D}\bar{\chi}\mathcal{D}\chi e^{-S_\Lambda} \ , \tag{4.28}$$

with the effective action

$$S_\Lambda = -N \ln \det\big(-\partial^2\big) \ . \tag{4.29}$$

Here is the place where the $-N$ comes into play. Would we have bosonic fields then the functional integral would give $\frac{1}{\det(-\partial^2)}$ instead of $\det(-\partial^2)$. The factor of 2 in Sp(2N) comes from the fact that χ are complex fields, whereas the O(N) model is formulated with N real fields [13].

Unsuccessful mapping of Sp(2N) to dS via Wick rotation

Now we would like to map the symmetries of Sp(2N) to the algebra (4.26) of the de Sitter group. We recognize that the analytical continuation $N \to -N$ was done between an O(N) model in Minkoswski spacetime and a Sp(2N) model in Euclidean spacetime.

Therefore we will first try to get the mapping through calculating the conformal algebra of the Sp(2N) as done for the O(N) model in chapter 4.1 but now in Euclidean space, which means we need a Wick rotation $\tau = it$. Instead of the Lagrangian in (2.43) we use the Euclidean Lagrangian

$$\mathcal{L}_E = \partial^\mu\bar{\chi}\partial_\mu\chi = \frac{\partial\bar{\chi}}{\partial\tau}\frac{\partial\chi}{\partial\tau} + \vec{\nabla}\bar{\chi}\vec{\nabla}\chi \ . \tag{4.30}$$

We remember the conjugate-momentum fields $\pi = -\partial_0\bar{\chi}$ and $\bar{\pi} = \partial_0\chi$ and that $\chi, \bar{\chi}, \pi, \bar{\pi}$ are Grassmann valued. As stated in chap. 2.3 the canonical anticommutation relation is $\{\chi(\vec{x},t), \pi(\vec{y},t)\} = \{\bar{\chi}(\vec{x},t), \bar{\pi}(\vec{y},t)\} = i\delta^2(\vec{x}-\vec{y}) \equiv i\delta(\vec{x}-\vec{y})$. We furthermore use $\{A,B\} = \{B,A\}$ as well as the identity

$$[AB, CD] = A\{B,C\}D - AC\{B,D\} + \{A,C\}DB - C\{A,D\}B \ . \tag{4.31}$$

The energy-momentum tensor of the Sp(2N) model reads (compare (2.52))

$$T_{\mu\nu} = \partial_\mu\bar{\chi}\partial_\nu\chi + \partial_\nu\bar{\chi}\partial_\mu\chi + \eta_{\mu\nu}\mathcal{L}_\chi \ . \tag{4.32}$$

As we have seen at (2.54) in chap. 2.3 in the Sp(2N) model, the energy-momentum tensor with two time indices equals to

$$T_{00}(x) = \bar{\pi}(x)\pi(x) + \vec{\nabla}\bar{\chi}(x)\vec{\nabla}\chi(x) \ . \tag{4.33}$$

Having a time and a space index we get

$$T_{0i}(x) = -\pi(x)\partial_i\chi(x) - \bar{\pi}(x)\partial_i\bar{\chi}(x) \tag{4.34}$$

which upon integration leads to the physical momentum P_i.

If we would now identify the conformal algebra, we would get the same algebra (4.17) as for the O(N) model. Performing the Wick rotation on (4.32), (4.33) and (4.34) with $\pi_\tau = -\frac{\partial\bar{\chi}}{\partial\tau}$ and $\bar{\pi}_\tau = \frac{\partial\chi}{\partial\tau}$ we have

$$T^E_{00}(x) = -\frac{\partial\bar{\chi}}{-i\partial\tau}\frac{\partial\chi}{-i\partial\tau} - \frac{\partial\chi}{-i\partial\tau}\frac{\partial\bar{\chi}}{-i\partial\tau} + \mathcal{L}_E = -\bar{\pi}(x)\pi(x) + \vec{\nabla}\bar{\chi}(x)\vec{\nabla}\chi(x) \tag{4.35}$$

and

$$T^E_{0i}(x) = \frac{\partial\bar{\chi}}{-i\partial\tau}\partial_i\chi - \frac{\partial\chi}{-i\partial\tau}\partial_i\bar{\chi} = -i\pi_\tau(x)\partial_i\chi(x) - i\bar{\pi}_\tau(x)\partial_i\bar{\chi}(x)\ . \tag{4.36}$$

We get the following building blocks in the Euclidean Sp(2N) model (with $T^E \equiv T$):

$$\begin{aligned}
[T_{0i}(x), T_{0j}(y)] &= \Big[i\pi(x)\partial_i\chi(x), i\pi(y)\partial_j\chi(y)\Big] + \Big[i\bar{\pi}(x)\partial_i\bar{\chi}(x), i\bar{\pi}(y)\partial_j\bar{\chi}(y)\Big] \\
&= -\pi(x)\Big\{\partial_i\chi(x), \pi(y)\Big\}\partial_j\chi(y) + \pi(y)\Big\{\pi(x), \partial_j\chi(y)\Big\}\partial_i\chi(x) \\
&- \bar{\pi}(x)\Big\{\partial_i\bar{\chi}(x), \bar{\pi}(y)\Big\}\partial_j\bar{\chi}(y) + \bar{\pi}(y)\Big\{\bar{\pi}(x), \partial_j\bar{\chi}(y)\Big\}\partial_i\bar{\chi}(x) \\
&= -i\Big(\pi(x)\frac{\partial}{\partial y^j}\chi(y) + \bar{\pi}(x)\frac{\partial}{\partial y^j}\bar{\chi}(y)\Big)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}) \\
&\quad + i\Big(\pi(y)\frac{\partial}{\partial x^i}\chi(x) + \bar{\pi}(y)\frac{\partial}{\partial x^i}\bar{\chi}(x)\Big)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y})\ ,
\end{aligned}$$

$$\begin{aligned}
[T_{00}(x), T_{0j}(y)] &= \Big[\pi(x)\bar{\pi}(x) - \vec{\nabla}\chi(x)\vec{\nabla}\bar{\chi}(x), -i\pi(y)\partial_j\chi(y) - i\bar{\pi}(y)\partial_j\bar{\chi}(y)\Big] \\
&= -i\Big[\pi(x)\bar{\pi}(x), \pi(y)\partial_j\chi(y)\Big] + i\Big[\vec{\nabla}\chi(x)\vec{\nabla}\bar{\chi}(x), \pi(y)\partial_j\chi(y)\Big] \\
&- i\Big[\pi(x)\bar{\pi}(x), \bar{\pi}(y)\partial_j\bar{\chi}(y)\Big] + i\Big[\vec{\nabla}\chi(x)\vec{\nabla}\bar{\chi}(x), \bar{\pi}(y)\partial_j\bar{\chi}(y)\Big] \\
&= i\pi(y)\Big\{\pi(x), \partial_j\chi(y)\Big\}\bar{\pi}(x) + i\Big\{\vec{\nabla}\chi(x), \pi(y)\Big\}\partial_j\chi(y)\vec{\nabla}\bar{\chi}(x) \\
&+ i\pi(x)\bar{\pi}(y)\Big\{\bar{\pi}(x), \partial_j\bar{\chi}(y)\Big\} + i\vec{\nabla}\chi(x)\Big\{\vec{\nabla}\bar{\chi}(x), \bar{\pi}(y)\Big\}\partial_j\bar{\chi}(y) \\
&= -\Big(\pi(y)\bar{\pi}(x) + \pi(x)\bar{\pi}(y)\Big)\frac{\partial}{\partial y^j}\delta(x-y) \\
&\quad - \Big(\frac{\partial}{\partial y^j}\chi(y)\vec{\nabla}_x\bar{\chi}(x) + \vec{\nabla}_x\chi(x)\frac{\partial}{\partial y^j}\bar{\chi}(y)\Big)\vec{\nabla}_x\delta(x-y)\ ,
\end{aligned}$$

$$\begin{aligned}
[T_{00}(x), T_{00}(y)] &= \Big[\pi(x)\bar{\pi}(x) - \vec{\nabla}\chi(x)\vec{\nabla}\bar{\chi}(x), \pi(y)\bar{\pi}(y) - \vec{\nabla}\chi(y)\vec{\nabla}\bar{\chi}(y)\Big] \\
&= -\Big[\vec{\nabla}\chi(x)\vec{\nabla}\bar{\chi}(x), \pi(y)\bar{\pi}(y)\Big] - \Big[\pi(x)\bar{\pi}(x), \vec{\nabla}\chi(y)\vec{\nabla}\bar{\chi}(y)\Big] \\
&= \vec{\nabla}\chi(x)\pi(y)\Big\{\vec{\nabla}\bar{\chi}(x), \bar{\pi}(y)\Big\} - \Big\{\vec{\nabla}\chi(x), \pi(y)\Big\}\bar{\pi}(y)\vec{\nabla}\bar{\chi}(x) \\
&+ \pi(x)\vec{\nabla}\chi(y)\Big\{\bar{\pi}(x), \vec{\nabla}\bar{\chi}(y)\Big\} - \Big\{\pi(x), \vec{\nabla}\chi(y)\Big\}\vec{\nabla}\bar{\chi}(y)\bar{\pi}(x) \\
&= i\Big(\vec{\nabla}_x\chi(x)\pi(y) - \bar{\pi}(y)\vec{\nabla}_x\bar{\chi}(x)\Big)\vec{\nabla}_x\delta(x-y) \\
&\quad + i\Big(\pi(x)\vec{\nabla}_y\chi(y) - \vec{\nabla}_y\bar{\chi}(y)\bar{\pi}(x)\Big)\vec{\nabla}_y\delta(x-y)\ .
\end{aligned}$$

Let us now calculate some commutators of the conformal algebra. We again start with the commutator $[D, P_\mu]$. As for the O(N) model we include the correction terms for the currents. Due to the $\bar{\chi}$ fields we have twice as many correction terms. For $\mu = j$ we get

$$
\begin{aligned}
[D, P_j] &= \int d^2\vec{x}d^2\vec{y}\ x^\rho[T_{0\rho}, T_{0j}] - \frac{1}{2}[\chi(x)\pi(x), \pi(y)\partial_j\chi(y)] - \frac{1}{2}[\bar{\chi}(x)\bar{\pi}(x), \bar{\pi}(y)\partial_j\bar{\chi}(y)] \\
&= \int d^2\vec{x}d^2\vec{y}\Big(x^0[T_{00}, T_{0j}] + x^i[T_{0i}, T_{0j}]\Big) + \frac{1}{2}\int d^2\vec{x}\Big(- \chi\partial_j\pi - \partial_j\chi\pi - \bar{\chi}\partial_j\bar{\pi} - \partial_j\bar{\chi}\bar{\pi}\Big) \ .
\end{aligned}
$$

As in the O(N) model the correction terms vanish and we continue with

$$
\begin{aligned}
[D, P_j] = \int d^2\vec{x}d^2\vec{y}\Bigg(& x^0\Big(- \pi(y)\bar{\pi}(x) - \pi(x)\bar{\pi}(y)\Big)\frac{\partial}{\partial y^j}\delta(x-y) \\
& - x^0\Big(\frac{\partial}{\partial y^j}\chi(y)\vec{\nabla}_x\bar{\chi}(x) + \vec{\nabla}_x\chi(x)\frac{\partial}{\partial y^j}\bar{\chi}(y)\Big)\vec{\nabla}_x\delta(x-y) \\
& - ix^i\Big(\pi(x)\frac{\partial}{\partial y^j}\chi(y) + \bar{\pi}(x)\frac{\partial}{\partial y^j}\bar{\chi}(y)\Big)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}) \\
& + ix^i\Big(\pi(y)\frac{\partial}{\partial x^i}\chi(x) + \bar{\pi}(y)\frac{\partial}{\partial x^i}\bar{\chi}(x)\Big)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y})\Bigg) \\
= \int d^2\vec{x}\Big(&x^0\partial_j\pi\bar{\pi} + x^0\pi\partial_j\bar{\pi} + x^0\partial_j\chi\vec{\nabla}\vec{\nabla}\bar{\chi} + x^0\vec{\nabla}\vec{\nabla}\chi\partial_j\bar{\chi} + i\pi\partial_j\chi + i\bar{\pi}\partial_j\bar{\chi}\Big) \\
= \int d^2\vec{x}\Big(&x^0\partial_j(\pi\bar{\pi}) - x^0\partial_j(\vec{\nabla}\chi\vec{\nabla}\bar{\chi}) + i\pi\partial_j\chi + i\bar{\pi}\partial_j\bar{\chi}\Big) = \int d^2\vec{x}\Big(i\pi\partial_j\chi + i\bar{\pi}\partial_j\bar{\chi}\Big) \\
= - \int & d^2\vec{x}T_{0j} = -P_j \ .
\end{aligned}
$$

For $\mu = 0$ we obtain

$$\begin{aligned}
[D, P_0] &= \int d^2\vec{x}d^2\vec{y}\ x^\rho[T_{0\rho}, T_{00}] + \frac{1}{2}\Big([\chi(x)\pi(x), \bar{\pi}(y)\pi(y)] + [\chi(x)\pi(x), \vec{\nabla}\bar{\chi}(y)\vec{\nabla}\chi(y)] \\
&\qquad + [\bar{\chi}(x)\bar{\pi}(x), \bar{\pi}(y)\pi(y)] + [\bar{\chi}(x)\bar{\pi}(x), \vec{\nabla}\bar{\chi}(y)\vec{\nabla}\chi(y)]\Big) \\
&= \int d^2\vec{x}d^2\vec{y}\Big(x^0[T_{00}, T_{00}] + x^i[T_{0i}, T_{00}]\Big) \\
&\quad + \frac{1}{2}\Big(- \bar{\pi}(y)\{\chi(x), \pi(y)\}\pi(x) - \chi(x)\vec{\nabla}\bar{\chi}(y)\{\pi(x), \vec{\nabla}\chi(y)\} \\
&\qquad + \{\bar{\chi}(x), \bar{\pi}(y)\}\pi(y)\bar{\pi}(x) + \bar{\chi}(x)\{\bar{\pi}(x), \vec{\nabla}\bar{\chi}(y)\}\vec{\nabla}\chi(y)\Big) \\
&= \int d^2\vec{x}d^2\vec{y}\Big(- ix^0\Big(- \vec{\nabla}_x\chi(x)\pi(y) + \bar{\pi}(y)\vec{\nabla}_x\bar{\chi}(x)\Big)\vec{\nabla}_x\delta(x-y) \\
&\qquad - ix^0\Big(- \pi(x)\vec{\nabla}_y\chi(y) + \vec{\nabla}_y\bar{\chi}(y)\bar{\pi}(x)\Big)\vec{\nabla}_y\delta(x-y) \\
&\qquad + x^i\Big(\pi(x)\bar{\pi}(y) + \pi(y)\bar{\pi}(x)\Big)\frac{\partial}{\partial x^i}\delta(x-y) \\
&\qquad + x^i\Big(- \frac{\partial}{\partial x^i}\chi(x)\vec{\nabla}_y\bar{\chi}(y) - \vec{\nabla}_y\chi(y)\frac{\partial}{\partial x^i}\bar{\chi}(x)\Big)\vec{\nabla}_y\delta(x-y)\Big) + \int d^2\vec{x}\Big(\pi\bar{\pi} - \vec{\nabla}\chi\vec{\nabla}\bar{\chi}\Big) \\
&= \int d^2\vec{x}\Big(- ix^0\vec{\nabla}\vec{\nabla}\chi\pi + ix^0\bar{\pi}\vec{\nabla}\vec{\nabla}\bar{\chi} - ix^0\pi\vec{\nabla}\vec{\nabla}\chi + ix^0\vec{\nabla}\vec{\nabla}\bar{\chi}\bar{\pi} \\
&\qquad - 2\pi\bar{\pi} - x^i\partial_i\pi\bar{\pi} - 2\pi\bar{\pi} - x^i\pi\partial_i\bar{\pi} + x^i\partial_i\chi\vec{\nabla}\vec{\nabla}\bar{\chi} + x^i\vec{\nabla}\vec{\nabla}\chi\partial_i\bar{\chi} + \pi\bar{\pi} - \vec{\nabla}\chi\vec{\nabla}\bar{\chi}\Big) \\
&= \int d^2\vec{x}\Big(- 4\pi\bar{\pi} - x^i\partial_i(\pi\bar{\pi}) - \vec{\nabla}x^i\partial_i\chi\vec{\nabla}\bar{\chi} - x^i\partial_i\vec{\nabla}\chi\vec{\nabla}\bar{\chi} - \vec{\nabla}x^i\vec{\nabla}\chi\partial_i\bar{\chi} - x^i\vec{\nabla}\chi\partial_i\vec{\nabla}\bar{\chi} + \pi\bar{\pi} - \vec{\nabla}\chi\vec{\nabla}\bar{\chi}\Big) \\
&= \int d^2\vec{x}\Big(- 4\pi\bar{\pi} + 2\pi\bar{\pi} - \vec{\nabla}\chi\vec{\nabla}\bar{\chi} - x^i\partial_i(\vec{\nabla}\chi\vec{\nabla}\bar{\chi}) - \vec{\nabla}\chi\vec{\nabla}\bar{\chi} + \pi\bar{\pi} - \vec{\nabla}\chi\vec{\nabla}\bar{\chi}\Big) \\
&= \int d^2\vec{x}\Big(- \pi\bar{\pi} - \vec{\nabla}\chi\vec{\nabla}\bar{\chi}\Big) = - \int d^2\vec{x}T_{00} = -P_0\ .
\end{aligned}$$

Combining the results for $\mu = j$ and $\mu = 0$ we have

$$[D, P_\mu] = -P_\mu\ , \tag{4.37}$$

which is the same as (4.15) just without the negativ imaginary unit $-i$.

Let us continue with the commutator $[K_\mu, P_\nu]$. For $\mu = i$ and $\nu = j$ we get

$$
\begin{aligned}
[K_i, P_j] = &\int d^2\vec{x}d^2\vec{y}\Big(x^2[T_{0i}, T_{0j}] - 2x_ix^\sigma[T_{0\sigma}, T_{0j}] - x_i[\chi(x)\pi(x), \pi(y)\partial_j\chi(y)] - x_i[\bar{\chi}(x)\bar{\pi}(x), \bar{\pi}(y)\bar{\partial}_j\chi(y)] \\
&- \frac{1}{2}\delta_i^0[\chi(x)^2, \pi(y)\partial_j\chi(y)] - \frac{1}{2}\delta_i^0[\bar{\chi}(x)^2, \bar{\pi}(y)\partial_j\bar{\chi}(y)]\Big) \\
= &\int d^2\vec{x}d^2\vec{y}\Big(x^2[T_{0i}, T_{0j}] - 2x_ix^0[T_{00}, T_{0j}] - 2x_ix^k[T_{0k}, T_{0j}]\Big) + \int d^2\vec{x}\Big(-\delta_i^0\chi\partial_j\chi - \delta_i^0\bar{\chi}\partial_j\bar{\chi}\Big) \\
= &\int d^2\vec{x}d^2\vec{y}\Big(-ix^2\Big(\pi(x)\frac{\partial}{\partial y^j}\chi(y) + \bar{\pi}(x)\frac{\partial}{\partial y^j}\bar{\chi}(y)\Big)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}) \\
&\quad + ix^2\Big(\pi(y)\frac{\partial}{\partial x^i}\chi(x) + \bar{\pi}(y)\frac{\partial}{\partial x^i}\bar{\chi}(x)\Big)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y}) \\
&\quad + 2x_ix^0\Big(\pi(y)\bar{\pi}(x) + \pi(x)\bar{\pi}(y)\Big)\frac{\partial}{\partial y^j}\delta(x-y) \\
&\quad + 2x_ix^0\Big(\frac{\partial}{\partial y^j}\chi(y)\vec{\nabla}_x\bar{\chi}(x) + \vec{\nabla}_x\chi(x)\frac{\partial}{\partial y^j}\bar{\chi}(y)\Big)\vec{\nabla}_x\delta(x-y) \\
&\quad + i2x_ix^k\Big(\pi(x)\frac{\partial}{\partial y^j}\chi(y) + \bar{\pi}(x)\frac{\partial}{\partial y^j}\bar{\chi}(y)\Big)\frac{\partial}{\partial x^k}\delta(\vec{x}-\vec{y}) \\
&\quad - i2x_ix^k\Big(\pi(y)\frac{\partial}{\partial x^k}\chi(x) + \bar{\pi}(y)\frac{\partial}{\partial x^k}\bar{\chi}(x)\Big)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y})\Big) + \int d^2\vec{x}\Big(\eta_{ij}\chi\pi + \eta_{ij}\bar{\chi}\bar{\pi}\Big) \\
= &\int d^2\vec{x}\Big(i2x_i\pi\partial_j\chi + ix^2\partial_i\pi\partial_j\chi + i2x_i\bar{\pi}\partial_j\bar{\chi} + ix^2\partial_i\bar{\pi}\partial_j\bar{\chi} - ix^2\partial_j\pi\partial_i\chi - ix^2\partial_j\bar{\pi}\partial_i\bar{\chi} \\
&\quad - 2x_ix^0\partial_j\pi\bar{\pi} - 2x_ix^0\pi\partial_j\bar{\pi} \\
&\quad - 2\vec{\nabla}x_ix^0\partial_j\chi\vec{\nabla}\bar{\chi} - 2x_ix^0\partial_j\chi\vec{\nabla}\vec{\nabla}\bar{\chi} - 2\vec{\nabla}x_ix^0\vec{\nabla}\chi\partial_j\bar{\chi} - 2x_ix^0\vec{\nabla}\vec{\nabla}\chi\partial_j\bar{\chi} \\
&\quad - i2x_i\pi\partial_j\chi - i2x_i\pi\partial_j\chi - i2x_ix^k\partial_k\pi\partial_j\chi - i2x_i\bar{\pi}\partial_j\bar{\chi} - i2x_i\bar{\pi}\partial_j\bar{\chi} \\
&\quad - i2x_ix^k\partial_k\bar{\pi}\partial_j\bar{\chi} + i2x_ix^k\partial_j\pi\partial_k\chi + i2x_ix^k\partial_j\bar{\pi}\partial_k\bar{\chi} + \eta_{ij}\chi\pi + \eta_{ij}\bar{\chi}\bar{\pi}\Big) \\
= &\int d^2\vec{x}\Big(-i2x_i\pi\partial_j\chi - i2x_i\bar{\pi}\partial_j\bar{\chi} + i2x_j\pi\partial_i\chi + i2x_j\bar{\pi}\partial_i\bar{\chi} \\
&+ 2\eta_{ij}x^0\pi\bar{\pi} - 2x^0\partial_j\chi\partial_i\bar{\chi} - 2\eta_{ij}x^0\vec{\nabla}\chi\vec{\nabla}\bar{\chi} + 2\vec{\nabla}x_ix^0\partial_j\chi\vec{\nabla}\bar{\chi} - 2x^0\partial_i\chi\partial_j\bar{\chi} + 2\vec{\nabla}x_ix^0\vec{\nabla}\chi\partial_j\bar{\chi} \\
&- i2x_i\pi\partial_j\chi - i2x_i\bar{\pi}\partial_j\bar{\chi} - i2\eta_{ij}x^k\pi\partial_k\chi - i2\eta_{ij}x^k\bar{\pi}\partial_k\bar{\chi} + i2x_i\pi\partial_j\chi + i2x_i\bar{\pi}\partial_j\bar{\chi} + \eta_{ij}\chi\pi + \eta_{ij}\bar{\chi}\bar{\pi}\Big) \\
= &\int d^2\vec{x}\Big(2\eta_{ij}x^0\Big(-\pi\bar{\pi} - \vec{\nabla}\chi\vec{\nabla}\bar{\chi}\Big) + \eta_{ij}\chi\pi + \eta_{ij}\bar{\chi}\bar{\pi} - i2\eta_{ij}x^k\Big(\pi\partial_k\chi + \bar{\pi}\partial_k\bar{\chi}\Big) \\
&\quad - i2x_i\Big(\pi\partial_j\chi + \bar{\pi}\partial_j\bar{\chi}\Big) + i2x_j\Big(\pi\partial_i\chi + \bar{\pi}\partial_i\bar{\chi}\Big)\Big) \\
= &\, 2\eta_{ij}D + 2L_{ij} \ .
\end{aligned}
$$

For the other possibilities of μ and ν we get $[K_i, P_0] = 2L_{i0}$, $[K_0, P_j] = 2L_{0j}$ and $[K_0, P_0] = -2D$. Combining these four result we obtain

$$[K_\mu, P_\nu] = 2\eta_{\mu\nu}D + 2L_{\mu\nu} \tag{4.38}$$

which again lacks the $-i$ compared to (4.16). Unfortunately the sign is not flipped compared to the other eight commutators as is would be necessary for the correspondence with the de Sitter conformal algebra (4.26). In sum all the nine commutators are the same as in the conformal algebra of the O(N) model (4.17) just without the $-i$.

Mapping of Sp(2N) to de Sitter spacetime

In order to find an adequate mapping of the Sp($2N$) model to the de Sitter conformal algebra we will now have a look at the parametrizations of both the AdS and the dS spacetime in chapter 2.1. There is hidden something more behind this apparently simple mapping between the Poincaré coordinates of AdS and dS via a double Wick rotation (2.1).

Instead of relating the Poincaré coordinates, we would like to link both parametrizations to each other. For AdS_4 with $d = 3$ we have the hypersurface

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 - (X^3)^2 + (X^4)^2 = -L_{AdS}^2 \; . \tag{4.39}$$

The parametrization (2.3) to get the Poincaré patch with $i \in \{1,2\}$ and $\vec{x} = (x^1, x^2)$ reads

$$X^0 = \frac{z}{2} + \frac{1}{2z}(\vec{x}^2 - t^2 + L_{AdS}^2) \qquad X^i = \frac{L_{AdS} x^i}{z} \tag{4.40}$$

$$X^3 = \frac{L_{AdS} t}{z} \qquad X^4 = \frac{z}{2} + \frac{1}{2z}(\vec{x}^2 - t^2 - L_{AdS}^2) \; . \tag{4.41}$$

We use the Wick rotations

$$L_{AdS} = -iL_{dS} \; , \; z = -i\tau \; , \; t \equiv x^0 = ix^3 \; , \; x^i = i\vartheta^i \; , \; \vec{x} = i\vec{\vartheta} \tag{4.42}$$

where $\vec{\vartheta} = (\vartheta^1, \vartheta^2)$. With this the parametrization (4.40) and (4.41) gets

$$X^0 = -i\left(\frac{\tau}{2} - \frac{1}{2\tau}\Big(-\vec{\vartheta}^2 + (x^3)^2 - L_{dS}^2\Big)\right) \qquad X^i = i\frac{L_{dS}(x^0)^i}{\tau} \tag{4.43}$$

$$X^3 = i\frac{L_{dS} x^3}{\tau} \qquad X^4 = -i\left(\frac{\tau}{2} - \frac{1}{2\tau}\Big(-\vec{\vartheta}^2 + (x^3)^2 + L_{dS}^2\Big)\right) \; . \tag{4.44}$$

Apart from the i this resembles the de Sitter Poincaré parametrization (2.7)

$$X^0 = \frac{\tau}{2} - \frac{1}{2\tau}(\vec{x}^2 + L^2) \; , \; X^i = \frac{Lx^i}{\tau} \; , \; X^3 = \frac{Lx^3}{\tau} \; , \; X^4 = \frac{\tau}{2} - \frac{1}{2\tau}(\vec{x}^2 - L^2) \tag{4.45}$$

if in (4.43) and (4.44) we have the metric $\eta_{\mu\nu} = \mathrm{diag}(-1,-1,1)$ with $\mu,\nu \in \{1,2,3\}$ (not {0,1,2}) and $\vec{x} = (-\vartheta^1, -\vartheta^2, x^3)$ and further interchange X^0 with $-X^4$. For the gravity side metric we therefore have $\eta_{mn} = \mathrm{diag}(-1,-1,-1,1)$ with $m,n \in \{0,1,2,3\}$. As ϑ^1 and ϑ^2 have opposite sign as x^3 in $\vec{x}$, both ϑ^1 and ϑ^2 could be interpreted as some kind of negative space coordinates.

On the field theory side we have to realize that we also have to transform the four conformal generators. The four generators expressed with coordinates x^μ with upper indices are

$$P_\mu = \int d^2\vec{x}\; T^0{}_\mu \qquad L_{\nu\rho} = \int d^2\vec{x}\; (x_\nu T^0{}_\rho - x_\rho T^0{}_\nu) = \int d^2\vec{x}\; (\eta_{\mu\nu} x^\mu T^0{}_\rho - \eta_{\mu\rho} x^\mu T^0{}_\nu)$$

$$D = \int d^2\vec{x}\; x^\rho T^0{}_\rho \qquad K_\mu = \int d^2\vec{x}\; (x^2 T^0{}_\mu - 2x_\mu x^\sigma T^0{}_\sigma) = \int d^2\vec{x}\; (\eta_{\rho\nu} x^\rho x^\nu T^0{}_\mu - 2\eta_{\rho\mu} x^\rho x^\sigma T^0{}_\sigma) \; .$$

Using the Wick rotations (4.42) we find

$$d^2\vec{x} \mapsto -d^2\vec{\vartheta} \qquad , \qquad x^\mu = (\vartheta, \vec{x}) \mapsto (ix^3, i\vec{\vartheta}) = ix^\mu \tag{4.46}$$

and furthermore we shift the 3D metric $\eta_{\mu\nu} = \mathrm{diag}(-1,-1,1)$ with $\mu,\nu \in \{1,2,3\}$ to the 3D metric $\delta_{\mu\nu} = \mathrm{diag}(-1,-1,-1)$ with $\mu,\nu \in \{0,1,2\}$. This notion of negative Euclidean spacetime is not trivial, as according to [10, 11] it allows us to find a non-compact complete manifold of positive curvature which corresponds to dS_4. Normally, Euclideanization is done by complexifying time. This modifies the geometry in the time direction and leaves the spatial directions relatively unscathed, hence Euclidean methods work fine in AdS/CFT. As the boundary of AdS lies at spatial infinity, complexifying time does not disturb the structure of the spacetime too drastically from the holographic point of view. Since, on the other hand, the de Sitter boundary is located at temporal infinity, it is clear that the traditional Euclideanization procedure is not fully adequate for asymptotic de Sitter spacetimes. We need to Euclideanize in a way which changes space instead of time. That is, instead of mapping the signature from $(-++)$ to $(+++)$, we map it to $(---)$, leaving time unchanged and complexifying space [11].

The generators then with (4.46) and $\delta_{\mu\nu} = \mathrm{diag}(-1,-1,-1)$ get

$$\begin{aligned}
&\tilde{P}_\mu = -\int d^2\vec{\vartheta}\; T^0{}_\mu = -P_\mu \qquad \tilde{L}_{\nu\rho} = -\tilde{\tilde{L}}_{\nu\rho} = \int d^2\vec{\vartheta}\; (\delta_{\mu\nu} i x^\mu T^0{}_\rho - \delta_{\mu\rho} i x^\mu T^0{}_\nu) = -iL_{\nu\rho} \\
&\tilde{D} = -i\int d^2\vec{\vartheta}\; x^\rho T^0{}_\rho = -iD \qquad \tilde{K}_\mu = -\int d^2\vec{\vartheta}\; (\delta_{\rho\nu} i x^\rho i x^\nu T^0{}_\mu - 2\delta_{\rho\mu} i x^\rho i x^\sigma T^0{}_\sigma) = -K_\mu \;.
\end{aligned} \tag{4.47}$$

We have set $\tilde{L}_{\nu\rho} = -\tilde{\tilde{L}}_{\nu\rho}$ due to $[\hat{L}_{\mu\nu}, \phi(0)] = -\mathcal{L}_{\mu\nu}\phi(0) \equiv \tilde{\mathcal{L}}_{\mu\nu}\phi(0)$. $\hat{L}_{\mu\nu}$ is the operator acting on the Hilbert space of quantum fields, whereas $\mathcal{L}_{\mu\nu}$ forms a representation of the Lorentz algebra [2].

With the transformed generators we obtain

$$\begin{aligned}
[\tilde{D}, \tilde{P}_\mu] &= i[D, P_\mu] = -iP_\mu = i\tilde{P}_\mu \\
[\tilde{D}, \tilde{K}_\mu] &= i[D, K_\mu] = iK_\mu = -i\tilde{K}_\mu \\
[\tilde{K}_\mu, \tilde{P}_\nu] &= [K_\mu, P_\nu] = 2\eta_{\mu\nu} D + 2L_{\mu\nu} = 2i\eta_{\mu\nu}\tilde{D} + 2i\tilde{L}_{\mu\nu} \\
[\tilde{L}_{\mu\nu}, \tilde{L}_{\rho\sigma}] &= -[L_{\mu\nu}, L_{\rho\sigma}] = -\eta_{\mu\sigma} L_{\nu\rho} + \dots = -i\eta_{\mu\sigma}\tilde{L}_{\nu\rho} + \dots \;.
\end{aligned} \tag{4.48}$$

We see that all but one commutators are the same as in the O(N) model conformal algebra (4.17), except for a minus factor. Only the commutator $[\tilde{K}_\mu, \tilde{P}_\nu]$ has a switched sign in relation to the other eight commutators. Thus it corresponds to the algebra (4.26) of the de Sitter group. We have found the mapping between the algebra of the Sp(2N) model and the algebra of the de Sitter group. However, we see that for the mapping of the symmetries we did not need the property that the fields are Grassmann valued. This seems only to be necessary for the dynamics and the correlators, as stated above.

Expressed for the conformal algebra in the higher spin gravity with oscillators as well as with the two component spinors (twistors), we can construct the following generators out of (3.16) in order to fit to the algebra of the de Sitter spacetime

$$\begin{aligned}
\hat{D} &= -iD = -\frac{i}{2}y^{+\alpha}y^{-}_{\alpha} = \frac{i}{4}y_{\alpha}\bar{y}^{\alpha} = \frac{i}{4}\epsilon^{\alpha\beta}y_{\alpha}\bar{y}_{\beta} \ , \\
\hat{P}_{\alpha\beta} &= -P_{\alpha\beta} = -iy^{-}_{\alpha}y^{-}_{\beta} = -\frac{i}{4}\Big(\bar{y}_{\alpha}\bar{y}_{\beta} - iy_{\alpha}\bar{y}_{\beta} - i\bar{y}_{\alpha}y_{\beta} - y_{\alpha}y_{\beta}\Big) = -\hat{K}_{\alpha\beta}(y \leftrightarrow \bar{y}) \ , \\
\hat{L}_{\alpha\beta} &= -iL_{\alpha\beta} = \epsilon_{\delta\alpha}\hat{L}^{\delta}{}_{\beta} = -i\epsilon_{\delta\alpha}\Big(y^{+\delta}y^{-}_{\beta} - \frac{1}{2}\delta^{\delta}_{\beta}y^{+\gamma}y^{-}_{\gamma}\Big) = -iy^{+}_{\alpha}y^{-}_{\beta} + \frac{i}{4}\epsilon_{\delta\alpha}\delta^{\delta}_{\beta}y^{\gamma}\bar{y}_{\gamma} \\
&= -\frac{i}{4}(y_{\alpha} - i\bar{y}_{\alpha})(\bar{y}_{\beta} - iy_{\beta}) + \frac{i}{4}\epsilon_{\delta\alpha}\delta^{\delta}_{\beta}\epsilon^{\gamma\epsilon}y_{\epsilon}\bar{y}_{\gamma} \\
&= -\frac{i}{4}(y_{\alpha}\bar{y}_{\beta} - i\bar{y}_{\alpha}\bar{y}_{\beta} - iy_{\alpha}y_{\beta} - \bar{y}_{\alpha}y_{\beta}) + \frac{i}{4}(\delta^{\gamma}_{\beta}\delta^{\epsilon}_{\alpha} - \delta^{\epsilon}_{\beta}\delta^{\gamma}_{\alpha})y_{\epsilon}\bar{y}_{\gamma} = -\frac{1}{4}(\bar{y}_{\alpha}\bar{y}_{\beta} + y_{\alpha}y_{\beta}) \ , \\
\hat{K}_{\alpha\beta} &= -K_{\alpha\beta} = iy^{+}_{\alpha}y^{+}_{\beta} = \frac{i}{4}\Big(y_{\alpha}y_{\beta} - i\bar{y}_{\alpha}y_{\beta} - iy_{\alpha}\bar{y}_{\beta} - \bar{y}_{\alpha}\bar{y}_{\beta}\Big) \ .
\end{aligned} \tag{4.49}$$

However, we just reflected upon the symmetry relations of the duality between the linear Sp($2N$) model and a linearized version of de Sitter spacetime. The search for a complete higher spin theory in de Sitter spacetime with its dynamics which is linked to a Sp($2N$) model has still to be done.

5 Conclusion

In this thesis we dealt with AdS/CFT and dS/CFT dualities in the framework of the higher spin gravity. The main aim was to find a mapping between a conformal algebra and the de Sitter spacetime. This duality is of high importance as, due to the fact that the cosmological constant seems to be positive, our universe is rather asymptotically de Sitter spacetime than Anti-de Sitter. Recent developments showed that the Sp($2N$) could proof as a probable candidate for the CFT dual to dS spacetime.

We first introduced the basic ingredients needed for the later calculation. These included the introduction to the two maximally symmetric spacetimes, AdS and dS. Then we presented the foundations of higher spin gravity in 4 dimensions which includes AdS_4 and plays an important role in the gauge/gravity duality as HS can help us to understand the first principles in the (A-)dS/CFT correspondence. Then we displayed the linear O(N) and Sp($2N$) models which are the field theories on the CFT side of the duality. The last part of the basic chapter introduced the renormalization group which plays a crucial role in describing field theories in dependence of the cut-off scale.

In chapter 3 we related the renormalization group equations of the linear O(N) model to the higher spin gravity in four dimensional Anti-de Sitter spacetime. We introduced two-component spinors as well as oscillators in order to formulate a conformal algebra. The classical renormalization group flow for traceless higher derivative couplings which are quadratic in the fields is identified with the higher spin equation of motion for the non-propagating sector on an AdS_4 background. Furthermore, gauge transformations and inhomogenities were discussed for higher spin fields within this duality.

In the final chapter we determined the conformal algebra of the O(N) model and recognized that it is necessary to add correction terms to the currents or to use the improved energy-momentum tensor in order for the theory to be invariant under scale transformations. Then, in the next subchapter, we compared the conformal algebra of AdS and dS spacetime and realized that one of the 9 commutators, namely $[K_\mu, P_\nu]$ has a flipped sign.

Finally we searched for a mapping between the conformal algebra of the Sp($2N$) model and de Sitter spacetime. The Sp($2N$) model with its anticommuting scalars is considered dual to de Sitter spacetime [8] due to dynamical considerations as an analogue to the conjectured O(N)-HS duality [14,15]. However, we found that a standard Wick rotation is not enough to map the symmetries of the Sp($2N$) model to the de Sitter spacetime. The important $[K_\mu, P_\nu]$ commutator did not flip its sign. In order to encounter an adequate mapping between the Sp($2N$) model to dS spacetime we had to relate the parametrizations of the AdS and dS spacetimes. In doing so, we find some very interesting transformations, which include the Wick rotations, but we also have to introduce two new coordinates ϑ^1 and ϑ^2. The signs of ϑ^1 and ϑ^2 are opposed to that of x^3, which could be interpreted in a way that ϑ^1 and ϑ^2 are some kind of negative space coordinates. With these transformations we reexpress our four conformal generators and receive the correct conformal algebra, where $[K_\mu, P_\nu]$ has flipped its sign as needed for the dS algebra and consequently differs from the O(N) model and the AdS algebra. This has led us to a spacetime with a notion of negative Euclidean signature, which at first glimpse appears trivial. However, it was already considered non-trivial at the beginning of this century [10, 11], due to the fact that the boundary of de Sitter spacetime is located at temporal infinity. This means that standard Euclideanization with complexifying time cannot be adequate and we rather have to complexify space, which leads to a signature $(- - -)$. This seems to be related to the fact that in the AdS/CFT correspondence the radial direction of AdS emerges holographically, while in dS/CFT it is the cosmological time direction of dS that emerges holographically. Finally, we used this new conformal algebra to adapt the algebra of the higher spin algebra based on the two-component spinors (twistors).

Of course, there are still many, many open questions, of which we will try to name a few. First of all, we just considered the symmetry relations of the $\mathrm{Sp}(2N)$ model and a linearized version of de Sitter spacetime. A complete higher spin theory in de Sitter spacetime with its dynamics and correct links to a $\mathrm{Sp}(2N)$ model is still unknown. As the higher spin gravity is considered the tensionless limit of string theory, which means that the string length can be seen as infinitely long, this non-locality would have serious consequences for the geometry of spacetime itself. Other open questions are what an action formalism with a Lagrangian could look like for the higher spin gravity and if the HS theory could be expressed without the auxiliary variables Z.

On the CFT side there are many questions as well. How could we interpret a $\mathrm{Sp}(2N)$ model with anticommuting scalars in the real world? Also, the form of the cut-off in the renormalization group equations still generates many unresolved questions [4].

Finally, of course the sketched mapping between the $\mathrm{Sp}(2N)$ model and de Sitter spacetime symmetries have to be analyzed in much more detail, especially the new coordinates ϑ^1 and ϑ^2 as well as the consequences of the notion of a negative Euclidean metric.

A Conventions and definitions

Index notation:

$(d+2)$ dimensional indices	$M, N, \ldots$
$(d+1)$ dimensional gravity (bulk) indices	$m, n, \ldots$
d dimensional field theory (boundary) indices	$\mu, \nu, \ldots$
$(d-1)$ dimensional spatial indices in field theory	$i, j, \ldots$

For the Lorentzian signature we use the mostly plus convention for the metric $(-++\cdots+)$.

For the dot-product, integrals, functionals and dimensions we followed the conventions and definitions of [44, 47]:

$$\phi \cdot \psi = \int d^d x \phi(x) \psi(x) \quad , \quad \phi(x) = \int \frac{d^d p}{(2\pi)^d} e^{ipx} \tilde{\phi}(p) \quad , \quad \phi(p) \equiv \tilde{\phi}(p) = \int d^d x e^{-ipx} \phi(x)$$

$$\int_x \equiv \int d^d x \quad , \quad \int_p \equiv \int \frac{d^d p}{(2\pi)^d} \quad , \quad \frac{\delta \phi(p)}{\delta \phi(q)} = \int d^d x e^{i(p-q)x} = (2\pi)^d \delta^d(p-q) \ ,$$

$$\frac{\delta}{\delta \phi(x)} = \int \frac{d^d p}{(2\pi)^d} e^{-ipx} \frac{\delta}{\delta \phi(p)} \quad , \quad \frac{\delta}{\delta \phi(p)} = \int d^d x e^{ipx} \frac{\delta}{\delta \phi(x)} \ .$$

This leads to

$$\begin{aligned} \phi \cdot \psi &= \int d^d x \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} e^{ipx} e^{iqx} \tilde{\phi}(p) \tilde{\psi}(q) = \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \left(\int d^d x e^{i(p+q)x} \right) \tilde{\phi}(p) \tilde{\psi}(q) \\ &= \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} (2\pi)^d \delta(p-q) \tilde{\phi}(p) \tilde{\psi}(q) = \int \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p) \tilde{\psi}(-p) \equiv \int_p \phi(p) \psi(-p) \end{aligned}$$

and

$$\phi \cdot \frac{\delta}{\delta \psi} = \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \left(\int d^d x e^{i(p+q)x} \right) \tilde{\phi}(p) \frac{\delta}{\delta \tilde{\psi}(p)} = \int_p \phi(p) \frac{\delta}{\delta \psi(p)} \ .$$

The spinors (twistors), charge conjugation matrix $\epsilon_{\alpha\beta}$, Pauli-matrices, Sigma-matrices and Gamma-matrices (with [38, 40, 51]) have the following properties[15]:

$$[y^-_\alpha, y^{+\beta}]_* = \delta^\beta_\alpha \quad , \quad \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \quad , \quad u_\alpha = u^\beta \epsilon_{\beta\alpha} \quad , \quad u^\alpha = \epsilon^{\alpha\beta} u_\beta \quad , \ \epsilon^{\alpha\beta} \epsilon_{\alpha\gamma} = \delta^\beta_\gamma$$

$$\epsilon_{\dot\alpha\dot\beta} = -\epsilon_{\dot\beta\dot\alpha} \quad , \quad u_{\dot\alpha} = u^{\dot\beta} \epsilon_{\dot\beta\dot\alpha} \quad , \quad u^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta} u_{\dot\beta} \quad , \ \epsilon^{\dot\alpha\dot\beta} \epsilon_{\dot\alpha\dot\gamma} = \delta^{\dot\beta}_{\dot\gamma} \quad , \quad \alpha, \beta, \ldots = 1, 2 \ , \ \dot\alpha, \dot\beta, \ldots = 1, 2$$

$$\epsilon_{\alpha\beta} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \quad , \ \epsilon_{12} = \epsilon^{12} = 1 \quad , \ \epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta}$$

$$\sigma_2 = \sigma_z = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix} \right) \quad , \quad (\sigma_2)_{\alpha\beta} = -i \epsilon_{\alpha\beta} \quad , \quad \epsilon^\alpha{}_\beta = -\delta^\alpha_\beta \quad , \quad (\sigma^\mu)_{\alpha\beta} (\sigma_\mu)^{\gamma\delta} = 2 \delta^\gamma_\alpha \delta^\delta_\beta$$

$$(\gamma_\mu)_\alpha{}^{\dot\beta} = i (\sigma_\mu)_\alpha{}^{\dot\beta} \quad , \quad (\gamma_\mu)_{\dot\alpha}{}^\beta = i (\sigma_\mu)^\beta{}_{\dot\alpha} \quad , \quad (\gamma_\mu)_\alpha{}^\beta = 0 \quad , \quad (\gamma_\mu)_{\dot\alpha}{}^{\dot\beta} = 0$$

[15] Note that in [26] the conventions for $\epsilon_{\alpha\beta}$ and for lowering/raising indices differ.

$$(\sigma_{\mu\nu})_{\alpha\beta} = \frac{1}{4}\Big((\sigma_\mu)_{\alpha\dot\alpha}(\sigma_\nu)_\beta{}^{\dot\alpha} - (\sigma_\nu)_{\alpha\dot\alpha}(\sigma_\mu)_\beta{}^{\dot\alpha}\Big) \quad , \quad (\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta} = \frac{1}{4}\Big((\sigma_\mu)_{\alpha\dot\alpha}(\sigma_\nu)^\alpha{}_{\dot\beta} - (\sigma_\nu)_{\alpha\dot\alpha}(\sigma_\mu)^\alpha{}_{\dot\beta}\Big)$$

$$(\sigma^\mu)_{\alpha\dot\alpha}(\sigma^\nu)^{\alpha\dot\beta} + (\mu \leftrightarrow \nu) = 2\delta^{\dot\beta}_{\dot\alpha}\eta^{\mu\nu} \quad , \quad (\bar\sigma_\mu)^{\dot\alpha\beta} \equiv (\sigma_\mu)^{\dot\alpha\beta} = (\sigma_\mu)_{\dot\gamma\delta}\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\beta\delta} \quad , \quad (\sigma^\mu)_{\alpha\dot\alpha}(\sigma^\nu)^{\alpha\dot\alpha} = 2\eta^{\mu\nu}$$

$$(\sigma^\mu)_{\alpha\dot\alpha}(\sigma^\nu)^{\beta\dot\alpha} + (\mu \leftrightarrow \nu) = 2\delta^\beta_\alpha\eta^{\mu\nu}$$

The 4D sigma matrices are:

$$(\sigma_m)_{\alpha\dot\beta} = (1, \vec\sigma)_{\alpha\dot\beta} \ , \ (\bar\sigma_m)^{\dot\alpha\beta} = (1, -\vec\sigma)^{\dot\alpha\beta} \ .$$

The 3D gamma matrices are obtained by deleting the matrix with space-time index $m = 2$:

$$(\gamma_\mu)_{\alpha\beta} = (1, \sigma_1, \sigma_3)_{\alpha\beta}$$

$x^2 = z$, $m = 0, 1, 2, 3$, $\mu = 0, 1, 3$ within the 4D $\leftrightarrow$ 3D, otherwise as usually $\mu = 0, 1, 2$ for $d = 3$.
$(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_{\beta\alpha}$ because $1, \sigma_1, \sigma_3$ are symmetric.

Y_A are complex two-component spinor (twistor) variables and C_{AB} is the 4D charge conjugation matrix with

$$Y_A = \begin{pmatrix} y_\alpha \\ \bar y_{\dot\alpha} \end{pmatrix} \quad , \quad Y_A * Y_B = Y_A Y_B + iC_{AB} \quad , \quad [Y_A, Y_B]_* = 2iC_{AB} \quad , \quad C_{AB} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot\alpha\dot\beta} \end{pmatrix} \ .$$

The relation between 3D gamma matrices and 4D sigma matrices are:

$$(\gamma_\mu)_{AB} = \begin{pmatrix} 0 & (\sigma_m)_{\alpha\dot\beta} \\ (\bar\sigma_m)_{\dot\alpha\beta} & 0 \end{pmatrix}$$

From

$$\gamma_{\nu\rho} = \frac{1}{4}[\gamma_\nu, \gamma_\rho] = \frac{1}{4}\Big(\gamma_\nu\gamma_\rho - \gamma_\rho\gamma_\nu\Big)$$

follows

$$\begin{aligned}(\gamma_{bc})_{AB} &= \frac{1}{4}\left(\begin{pmatrix} 0 & (\sigma_b)_{\alpha\dot\beta} \\ (\bar\sigma_b)_{\dot\alpha\beta} & 0 \end{pmatrix}\begin{pmatrix} 0 & (\sigma_c)_{\alpha\dot\beta} \\ (\bar\sigma_c)_{\dot\alpha\beta} & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\sigma_c)_{\alpha\dot\beta} \\ (\bar\sigma_c)_{\dot\alpha\beta} & 0 \end{pmatrix}\begin{pmatrix} 0 & (\sigma_b)_{\alpha\dot\beta} \\ (\bar\sigma_b)_{\dot\alpha\beta} & 0 \end{pmatrix}\right) \\ &= \frac{1}{4}\begin{pmatrix} (\sigma_b)_{\alpha\dot\beta}(\bar\sigma_c)_{\dot\alpha\beta} - (\sigma_c)_{\alpha\dot\beta}(\bar\sigma_b)_{\dot\alpha\beta} & 0 \\ 0 & (\bar\sigma_b)_{\dot\alpha\beta}(\sigma_c)_{\alpha\dot\beta} - (\bar\sigma_c)_{\dot\alpha\beta}(\sigma_b)_{\alpha\dot\beta} \end{pmatrix}\end{aligned}$$

and

$$(\gamma^a)^{\alpha\beta} = \epsilon^{abc}(\gamma_{bc})^{\alpha\beta} \quad \rightarrow \quad (\sigma^a)^{\alpha\dot\beta} = \epsilon^{abc}(\sigma_{bc})^{\alpha\beta} \quad , \quad (\sigma^a)^{\dot\alpha\beta} = \epsilon^{abc}(\bar\sigma_{bc})^{\dot\alpha\dot\beta} \ .$$

The Weyl (Moyal) star-product has the following properties (proofs in C.1):

$$(f * g)(y) = \frac{1}{(2\pi)^2}\int d^2u \ d^2v \ e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}} f(y+u)g(y+v) = f(y)e^{i\epsilon_{\alpha\beta}\frac{\overleftarrow{\partial}}{\partial y_\alpha}\frac{\overrightarrow{\partial}}{\partial y_\beta}}g(y)$$

$$y_\alpha * y_\beta = \frac{1}{(2\pi)^2}\int d^2u d^2v e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}}(y_\alpha + u_\alpha)(y_\beta + v_\beta) = y_\alpha \exp\left(i\frac{\overleftarrow{\partial}}{\partial y_\gamma}\epsilon_{\gamma\delta}\frac{\overrightarrow{\partial}}{\partial y_\delta}\right)y_\beta = y_\alpha y_\beta + i\epsilon_{\alpha\beta}$$

$$(f*g)(z) = \frac{1}{(2\pi)^2}\int d^2u\ d^2v\ e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}} f(z+u)g(z-v) = f(y)e^{-i\epsilon_{\alpha\beta}\frac{\overleftarrow{\partial}}{\partial y_\alpha}\frac{\overrightarrow{\partial}}{\partial y_\beta}} g(y)$$

$$z_\alpha * z_\beta = \frac{1}{(2\pi)^2}\int d^2ud^2ve^{iu^\gamma v^\delta\epsilon_{\gamma\delta}}(z_\alpha+u_\alpha)(z_\beta-v_\beta) = z_\alpha \exp\Big(-i\frac{\overleftarrow{\partial}}{\partial y_\gamma}\epsilon_{\gamma\delta}\frac{\overrightarrow{\partial}}{\partial y_\delta}\Big)z_\beta = z_\alpha z_\beta - i\epsilon_{\alpha\beta}$$

$$(f*g)(y,z) = \frac{1}{(2\pi)^2}\int d^2u\ d^2v\ e^{iu^\gamma v^\delta\epsilon_{\gamma\delta}} f(y+u,z+u)g(y+v,z-v) = f(y,z)e^{i\epsilon_{\alpha\beta}\left(\frac{\overleftarrow{\partial}}{\partial y_\alpha}+\frac{\overleftarrow{\partial}}{\partial z_\alpha}\right)\left(\frac{\overrightarrow{\partial}}{\partial y_\beta}-\frac{\overrightarrow{\partial}}{\partial z_\beta}\right)} g(y,z)$$

$$(f*g)*h = f*(g*h) \quad , \quad K*K = 1 \quad \text{proofs in C.1}$$

with commutator relations [39]

$$[y_\alpha, y_\beta]_* = 2i\epsilon_{\alpha\beta} \quad , \quad [\bar{y}_{\dot\alpha}, \bar{y}_{\dot\beta}]_* = 2i\epsilon_{\dot\alpha\dot\beta} \quad , \quad [y_\alpha, \bar{y}_{\dot\beta}]_* = 0 \quad , \quad [y_\alpha, z_\beta]_* = 0$$

$$[z_\alpha, z_\beta]_* = -2i\epsilon_{\alpha\beta} \quad , \quad [\bar{z}_{\dot\alpha}, \bar{z}_{\dot\beta}]_* = -2i\epsilon_{\dot\alpha\dot\beta} \quad , \quad [z_\alpha, \bar{z}_{\dot\beta}]_* = 0 \quad , \quad [\bar{y}_{\dot\alpha}, \bar{z}_{\dot\beta}]_* = 0$$

and

$$y_\alpha z^\alpha = \epsilon^{\alpha\beta}y_\alpha z_\beta = \epsilon^{\alpha\beta}z_\beta y_\alpha = -\epsilon^{\beta\alpha}z_\beta y_\alpha = -z_\beta y^\beta = -z_\alpha y^\alpha$$

and more commutator relations

$$\begin{aligned}
&[y_\alpha y_\beta, y_\gamma y_\delta]_* = 2i\epsilon_{\alpha\gamma}y_\beta y_\delta + 2i\epsilon_{\alpha\delta}y_\beta y_\gamma + 2i\epsilon_{\beta\gamma}y_\alpha y_\delta + 2i\epsilon_{\beta\delta}y_\alpha y_\gamma \\
&[y_\alpha y_\beta, y_\gamma \bar{y}_{\dot\delta}]_* = 2i\epsilon_{\alpha\gamma}y_\beta\bar{y}_{\dot\delta} + 2i\epsilon_{\beta\gamma}y_\alpha\bar{y}_{\dot\delta} \\
&[y_\alpha y_\beta, \bar{y}_{\dot\gamma}\bar{y}_{\dot\delta}]_* = 0\ .
\end{aligned}$$

Only if $f(y), g(y), h(y)$ are <u>linear in y</u> then Leibniz rule is valid (if not linear in y, then higher derivatives do not vanish) (proof in appendix C.1):

$$[y_\alpha y_\beta, y_\gamma]_* = y_\alpha[y_\beta, y_\gamma]_* + [y_\alpha, y_\gamma]_* y_\beta = 2i\epsilon_{\alpha\gamma}y_\beta + 2i\epsilon_{\beta\gamma}y_\alpha$$

Oscillators y^+_α and y^-_α are

$$y^+_\alpha = \frac{1}{2}(y_\alpha - i\bar{y}_\alpha)\ ,\ y^-_\alpha = \frac{1}{2}(\bar{y}_\alpha - iy_\alpha) \text{ thus } y_\alpha = y^+_\alpha + iy^-_\alpha\ ,\ \bar{y}_\alpha = y^-_\alpha + iy^+_\alpha$$

$$y^-_\alpha y^{+\alpha} = \frac{1}{2}y^\alpha\bar{y}_\alpha = -\frac{1}{2}y_\alpha\bar{y}^\alpha$$

$$y^\alpha\bar{y}_\alpha = \epsilon^{\alpha\beta}y_\beta\bar{y}_\alpha = -y_\beta\bar{y}^\beta = -\bar{y}^\alpha y_\alpha$$

and

$$y_\alpha y^\alpha + \bar{y}_\alpha\bar{y}^\alpha = 2i(y^-_\alpha y^{+\alpha} + y^+_\alpha y^{-\alpha}) = i(y^\alpha\bar{y}_\alpha + \bar{y}^\alpha y_\alpha) = 0\ .$$

B Basics on O(N) and Sp($2N$) groups

The set of elements $\{A, B, C, \dots\}$ forms a group G if it satisfies the following four axioms [33].
1. Among the elements there is an identity element I such that $AI = IA = A$.
2. The product AB gives another element C in the set. This means, the set is closed.
3. There exists an inverse element A^{-1} such that $A^{-1}A = AA^{-1} = I$.
4. The set is associative as $A(BC) = (AB)C$.

Among the continuous groups exist those orthogonal groups which are defined by the orthogonality condition $A^T A = I$. Usually these groups act on real number fields and are denoted $\mathrm{O}(N) \equiv \mathrm{O}(N, \mathbb{R})$. The (real) orthogonal group is a Lie group with the dimension $\frac{1}{2}N(N-1)$. $\mathrm{O}(N)$ characterizes the symmetry of the sphere, with N real coordinates, in the sense that the radius $\sum_{i=1}^{N} x_i^2$ is invariant under linear transformations with matrices from $\mathrm{O}(N)$. $\mathrm{O}(N)$ has two connected components, which are characterized by either $\det A = 1$ or $\det A = -1$. If $\det A = 1$ we have the subgroup $\mathrm{SO}(N)$, called special orthogonal group or rotation group. In the context of relativity we need to consider the pseudo-orthogonal groups $\mathrm{O}(N, M)$ which leave invariant

$$\sum_{\mu\nu} \eta^{\mu\nu} x_\mu x_\nu \tag{B.1}$$

with $\eta_{\mu\nu} = \delta_{\mu\nu}$ for $\mu = 1, \dots, N$ and $\eta_{\mu\nu} = -\delta_{\mu\nu}$ for $\mu = N+1, \dots, M$. Examples for special special-orthogonal groups are the Lorentz group $\mathrm{SO}(d, 1)$, the de Sitter group $\mathrm{SO}(d+1, 1)$ and the Anti-de Sitter group $\mathrm{SO}(d, 2)$ [32, 33].

Other continuous groups are the unitary groups $\mathrm{U}(N)$ over complex numbers with the condition $A^\dagger A = I$. They leave the quantity $\sum_{i=1}^{n} z_i \bar{z}_i$ invariant. A complex scalar field is invariant under the U(1) transformation $\phi(x) \mapsto \phi(x)' = e^{i\alpha}\phi(x)$, which is an example for an internal symmetry.

To define symplectic groups we divide the vectors $\vec{x}$ and $\vec{y}$ into two pieces,
$\vec{x} = (x_1, \dots, x_n; x'_1, \dots, x'_n)$, $\vec{y} = (y_1, \dots, y_n; y'_1, \dots, y'_n)$.
Symplectic groups $\mathrm{Sp}(2N, \mathbb{C})$ and $\mathrm{Sp}(2N, \mathbb{R})$ are defined as those groups that leave invariant

$$\sum_{i=1}^{N} (x_i y'_i - y_i x'_i) \tag{B.2}$$

where the vectors can be real or complex. $\mathrm{Sp}(2N, \mathbb{C})$ and $\mathrm{Sp}(2N, \mathbb{R})$ are Lie groups with dimension $2N(2N+1)$ respectively $N(2N+1)$. If we furthermore impose $A^\dagger A = I$, the group is called unitary symplectic $\mathrm{USp}(2N, \mathbb{C})$ which we will simply denote by $\mathrm{Sp}(2N)$ [32, 33]. Matrices of $\mathrm{Sp}(2N)$ are those elements A which obey

$$A^T \epsilon_N A = \epsilon_N \qquad \text{with} \qquad \epsilon_N = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} . \tag{B.3}$$

C Detailed calculations

C.1 Star product and spinors (twistors)

Proof of correspondence between integral and differential version of the Weyl (Moyal) star-product is shown in the following. For the case with the spinor (twistor) variable y, we start with

$$(f * g)(y) = \frac{1}{(2\pi)^2}\int d^2u\ d^2v\ e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}} f(y+u)g(y+v)$$

and now make a Fourier-transformation of $f(y+u)$ and $g(y+v)$

$$\begin{aligned}
(f * g)(y) &= \frac{1}{(2\pi)^2}\int d^2u d^2v d^2p d^2q e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}} \tilde{f}(p) e^{i(y_\alpha+u_\alpha)p^\alpha} \tilde{g}(q) e^{i(y_\alpha+v_\alpha)q^\alpha} \\
&= \frac{1}{(2\pi)^2}\int d^2v d^2p d^2q \tilde{f}(p)\tilde{g}(q) e^{iy_\alpha p^\alpha} e^{iy_\alpha q^\alpha} e^{iv_\alpha q^\alpha} \int d^2u \exp i(u^\gamma \epsilon_{\gamma\delta} v^\delta + u_\alpha p^\alpha) \\
&= \frac{1}{(2\pi)^2}\int d^2v d^2p d^2q \tilde{f}(p)\tilde{g}(q) e^{iy_\alpha p^\alpha} e^{iy_\alpha q^\alpha} e^{iv_\alpha q^\alpha} \int d^2u \exp i(-u^\alpha v_\alpha - u^\alpha p_\alpha) \\
&= \frac{1}{(2\pi)^2}\int d^2v d^2p d^2q \tilde{f}(p)\tilde{g}(q) e^{iy_\alpha p^\alpha} e^{iy_\alpha q^\alpha} e^{iv_\alpha q^\alpha} \int d^2u \exp iu^\alpha(-v_\alpha - p_\alpha) \\
&= \int d^2v d^2p d^2q \tilde{f}(p)\tilde{g}(q) e^{iy_\alpha p^\alpha} e^{iy_\alpha q^\alpha} e^{iv_\alpha q^\alpha} \delta(-v_\alpha - p_\alpha) \\
&= \int d^2p d^2q \tilde{f}(p)\tilde{g}(q) e^{iy_\alpha p^\alpha} e^{iy_\alpha q^\alpha} e^{-ip_\alpha q^\alpha} = \int d^2p d^2q \tilde{f}(p)\tilde{g}(q) e^{iy_\alpha p^\alpha} e^{iy_\alpha q^\alpha} e^{-i\epsilon_{\beta\alpha} p^\beta q^\alpha} \\
&= \int d^2p d^2q \tilde{f}(p)\tilde{g}(q) e^{iy_\alpha p^\alpha} e^{i\epsilon_{\alpha\beta} ip^\alpha iq^\beta} e^{iy_\alpha q^\alpha}
\end{aligned}$$

and finally we use $ip_\alpha = \partial_\alpha$ and get

$$(f * g)(y) = \int d^2p d^2q \tilde{f}(p) e^{iy_\alpha p^\alpha} e^{i\epsilon_{\alpha\beta} \frac{\overleftarrow{\partial}}{\partial y_\alpha} \frac{\overrightarrow{\partial}}{\partial y_\beta}} \tilde{g}(q) e^{iy_\alpha q^\alpha} = f(y) e^{i\epsilon_{\alpha\beta} \frac{\overleftarrow{\partial}}{\partial y_\alpha} \frac{\overrightarrow{\partial}}{\partial y_\beta}} g(y)\ .$$

For z (the auxiliary twistor variable) as the argument, it is the same calculation, we just have to replace v by $-v$

$$(f * g)(z) = \frac{1}{(2\pi)^2}\int d^2u\ d^2v\ e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}} f(z+u)g(z-v) = f(z) e^{-i\epsilon_{\alpha\beta} \frac{\overleftarrow{\partial}}{\partial z_\alpha} \frac{\overrightarrow{\partial}}{\partial z_\beta}} g(z)\ .$$

For the case with both arguments y and z we begin with

$$(f*g)(y,z) = \frac{1}{(2\pi)^2}\int d^2u\ d^2v\ e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}} f(y+u, z+u) g(y+v, z-v)\ ,$$

Fourier-transform $f(y+u, z+u)$ and $g(y+v, z-v)$ to get

$$\begin{aligned}
(f*g)(y,z) &= \frac{1}{(2\pi)^2}\int d^2u d^2v d^2p d^2r d^2q d^2s e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}} \tilde{f}(p,r) e^{i(y_\alpha+u_\alpha)p^\alpha} e^{i(z_\alpha+u_\alpha)r^\alpha} \tilde{g}(q,s) e^{i(y_\alpha+v_\alpha)q^\alpha} e^{i(z_\alpha-v_\alpha)s^\alpha} \\
&= \frac{1}{(2\pi)^2}\int d^2v d^2p d^2r d^2q d^2s \tilde{f}(p,r) e^{iy_\alpha p^\alpha} e^{iz_\alpha r^\alpha} \tilde{g}(q,s) e^{iy_\alpha q^\alpha} e^{iz_\alpha s^\alpha} \\
&\qquad e^{iv_\alpha(q^\alpha - s^\alpha)} \int d^2u \exp(i(u^\gamma \epsilon_{\gamma\delta} v^\delta + u_\alpha p^\alpha + u_\alpha r^\alpha)) \\
&= \frac{1}{(2\pi)^2}\int d^2v d^2p d^2r d^2q d^2s \tilde{f}(p,r) e^{iy_\alpha p^\alpha} e^{iz_\alpha r^\alpha} \tilde{g}(q,s) e^{iy_\alpha q^\alpha} e^{iz_\alpha s^\alpha} \\
&\qquad e^{iv_\alpha(q^\alpha - s^\alpha)} \int d^2u \exp(i(u^\alpha(-v_\alpha - p_\alpha - r_\alpha)) \\
&= \int d^2v d^2p d^2r d^2q d^2s \tilde{f}(p,r) e^{iy_\alpha p^\alpha} e^{iz_\alpha r^\alpha} \tilde{g}(q,s) e^{iy_\alpha q^\alpha} e^{iz_\alpha s^\alpha} e^{iv_\alpha(q^\alpha - s^\alpha)} \delta(-v_\alpha - (p_\alpha + r_\alpha)) \\
&= \int d^2p d^2r d^2q d^2s \tilde{f}(p,r) e^{iy_\alpha p^\alpha} e^{iz_\alpha r^\alpha} \tilde{g}(q,s) e^{iy_\alpha q^\alpha} e^{iz_\alpha s^\alpha} e^{i(p_\alpha + r_\alpha)(q^\alpha - s^\alpha)} \\
&= \int d^2p d^2r d^2q d^2s \tilde{f}(p,r) e^{iy_\alpha p^\alpha} e^{iz_\alpha r^\alpha} \tilde{g}(q,s) e^{iy_\alpha q^\alpha} e^{iz_\alpha s^\alpha} e^{i\epsilon_{\alpha\beta}(ip^\alpha iq^\beta - ip^\alpha is^\beta + ir^\alpha iq^\alpha - ir^\alpha is^\beta)}
\end{aligned}$$

and again we use $ip_\alpha = \partial_\alpha$ and receive

$$\begin{aligned}
(f*g)(y,z) &= f(y,z)\exp\left[i\epsilon_{\alpha\beta}\Big(\frac{\overleftarrow{\partial}}{\partial y_\alpha}\frac{\overrightarrow{\partial}}{\partial y_\beta} - \frac{\overleftarrow{\partial}}{\partial y_\alpha}\frac{\overrightarrow{\partial}}{\partial z_\beta} + \frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial y_\beta} - \frac{\overleftarrow{\partial}}{\partial z_\alpha}\frac{\overrightarrow{\partial}}{\partial z_\beta}\Big)\right] g(y,z) \\
&= f(y,z) e^{i\epsilon_{\alpha\beta}\left(\frac{\overleftarrow{\partial}}{\partial y_\alpha} + \frac{\overleftarrow{\partial}}{\partial z_\alpha}\right)\left(\frac{\overrightarrow{\partial}}{\partial y_\beta} - \frac{\overrightarrow{\partial}}{\partial z_\beta}\right)} g(y,z)\ .
\end{aligned}$$

Proof of associativity of the star product:

$$\begin{aligned}
(f*g)*h = (f(y)*g(y))*h(y) &= \left[\frac{1}{(2\pi)^2}\int d^2u\ d^2v\ e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}} f(y+u) g(y+v)\right] * h(y) \\
&= \frac{1}{(2\pi)^2}\int d^2w d^2x\ e^{iw_\alpha x^\alpha}\left[\frac{1}{(2\pi)^2}\int d^2u\ d^2v\ e^{iu_\alpha v^\alpha} f(y+u+w) g(y+v+w)\right] h(y+x) = \ [\text{FT}] \\
&= \frac{1}{(2\pi)^4}\int d^2w d^2x d^2u d^2v\ e^{iw_\alpha x^\alpha} e^{iu_\alpha v^\alpha} \tilde{f}(p) e^{i(y_\alpha+u_\alpha+w_\alpha)p^\alpha} \tilde{g}(q) e^{i(y_\alpha+v_\alpha+w_\alpha)q^\alpha} \tilde{h}(r) e^{i(y_\alpha+x_\alpha)r^\alpha} \\
&= \int d^2x d^2v\ \tilde{f}(p)\tilde{g}(q)\tilde{h}(r) e^{iy_\alpha p^\alpha} e^{iy_\alpha q^\alpha} e^{iv_\alpha q^\alpha} e^{iy_\alpha r^\alpha} e^{ix_\alpha r^\alpha} \delta(-v_\alpha - p_\alpha)\delta(-x_\alpha - (p_\alpha - q_\alpha)) \\
&= \int d^2x d^2v\ \tilde{f}(p)\tilde{g}(q)\tilde{h}(r) e^{iy_\alpha p^\alpha} e^{iy_\alpha q^\alpha} e^{-ip_\alpha q^\alpha} e^{iy_\alpha r^\alpha} e^{-ip_\alpha r^\alpha} e^{-iq_\alpha r^\alpha}
\end{aligned}$$

$$\begin{aligned}
f*(g*h) = f(y)*(g(y)*h(y)) &= f(y) * \left[\frac{1}{(2\pi)^2}\int d^2u\ d^2v\ e^{iu^\gamma v^\delta \epsilon_{\gamma\delta}} g(y+u) h(y+v)\right] \\
&= \frac{1}{(2\pi)^4}\int d^2w d^2x d^2u d^2v\ e^{iw_\alpha x^\alpha} e^{iu_\alpha v^\alpha} \tilde{f}(p) e^{i(y_\alpha+w_\alpha)p^\alpha} \tilde{g}(q) e^{i(y_\alpha+u_\alpha+x_\alpha)q^\alpha} \tilde{h}(r) e^{i(y_\alpha+v_\alpha+x_\alpha)r^\alpha} \\
&= \int d^2x d^2v\ \tilde{f}(p)\tilde{g}(q)\tilde{h}(r) e^{iy_\alpha p^\alpha} e^{iy_\alpha q^\alpha} e^{ix_\alpha q^\alpha} e^{iy_\alpha r^\alpha} e^{iv_\alpha r^\alpha} e^{ix_\alpha r^\alpha} \delta(-v_\alpha - q_\alpha)\delta(-x_\alpha - p_\alpha) \\
&= \int d^2x d^2v\ \tilde{f}(p)\tilde{g}(q)\tilde{h}(r) e^{iy_\alpha p^\alpha} e^{iy_\alpha q^\alpha} e^{-ip_\alpha q^\alpha} e^{iy_\alpha r^\alpha} e^{-ip_\alpha r^\alpha} e^{-iq_\alpha r^\alpha} = (f*g)*h
\end{aligned}$$

Proof that 1 is the unit element of the star product:

$$f(y) * 1 = \frac{1}{(2\pi)^2} \int d^2u d^2v \; e^{iu_\gamma v^\gamma} f(y+u) \cdot 1 = \frac{1}{(2\pi)^2} \int d^2u f(y+u) \int d^2v \exp(iv_\gamma(u^\gamma - 0))$$
$$= \int d^2u f(y+u) \delta(u) = f(y)$$

Proof that the inner Kleinian has the property $K * K = 1$:

$$K * K = e^{iz_\alpha y^\alpha} * e^{iz_\beta y^\beta} = \frac{1}{(2\pi)^2} \int d^2u \; d^2v \; e^{iu_\gamma v^\gamma} e^{i(z_\alpha + u_\alpha)(y^\alpha + u^\alpha)} e^{i(z_\beta - v_\beta)(y^\beta + v^\beta)}$$
$$= \frac{1}{(2\pi)^2} \int d^2u d^2v \; e^{i(u_\gamma v^\gamma + z_\alpha y^\alpha + z_\alpha u^\alpha + u_\alpha y^\alpha + u_\alpha u^\alpha + z_\beta y^\beta + z_\beta v^\beta - v_\beta y^\beta - v_\beta v^\beta)}$$
$$= \frac{1}{(2\pi)^2} \int d^2v \; e^{i(z_\alpha y^\alpha + z_\beta y^\beta + z_\beta v^\beta - v_\beta y^\beta - v_\beta v^\beta)} \int d^2u \; \exp(iu_\alpha(v^\alpha - z^\alpha + y^\alpha + u^\alpha))$$
$$= \frac{1}{(2\pi)^2} \int d^2v \; e^{i(z_\alpha y^\alpha + z_\beta y^\beta + z_\beta v^\beta - v_\beta y^\beta - v_\beta v^\beta)} \int d^2u \; \exp(iu_\alpha u^\alpha - iu_\alpha(z^\alpha - v^\alpha - y^\alpha))$$
$$= \frac{1}{(2\pi)^2} \int d^2v \exp(i(z_\alpha y^\alpha + z_\beta y^\beta + z_\beta v^\beta - v_\beta y^\beta - v_\beta v^\beta))(-2\pi i) \exp(i(z_\alpha - v_\alpha - y_\alpha)^2)$$
$$= -\frac{i}{2\pi} \int d^2v \exp(i(z_\alpha y^\alpha + z_\beta y^\beta + z_\beta v^\beta - v_\beta y^\beta - v_\beta v^\beta + z_\alpha z^\alpha + v_\alpha v^\alpha + y_\alpha y^\alpha$$
$$- z_\alpha v^\alpha - z_\alpha y^\alpha - v_\alpha z^\alpha + v_\alpha y^\alpha - y_\alpha z^\alpha + y_\alpha v^\alpha))$$

using $-v_\alpha z^\alpha = z_\alpha v^\alpha$, $-y_\alpha z^\alpha = z_\alpha y^\alpha$ and $-y_\alpha v^\alpha = v_\alpha y^\alpha$ we get

$$= -\frac{i}{2\pi} \int d^2v \exp\Big(- i(v_\beta v^\beta + v_\alpha v^\alpha) - iv_\beta(y^\beta + z^\beta) + i(z_\alpha y^\alpha + z_\beta y^\beta + z_\alpha z^\alpha + y_\alpha y^\alpha)\Big)$$
$$= \exp\Big(- i((y^\beta + z^\beta)^2 + i(z^\alpha + y^\alpha)^2\Big) = 1 \; .$$

With (2.17) we can show

$$[y_\alpha, y_\beta]_* = y_\alpha * y_\beta - y_\beta * y_\alpha = y_\alpha y_\beta + i\epsilon_{\alpha\beta} - y_\beta y_\alpha - i\epsilon_{\beta\alpha} = 2i\epsilon_{\alpha\beta} \; ,$$

and analogue for the barred terms.

Proof of Leibniz rule if $f(y), g(y), h(y)$ are <u>linear in y</u> (if not linear in y, then higher derivatives do not vanish and are not corresponding to Leibniz rule):

$$
\begin{aligned}
[y_\alpha y_\beta, y_\gamma]_* &= y_\alpha y_\beta y_\gamma + i\epsilon_{\delta\sigma}\frac{\partial}{\partial y_\delta}(y_\alpha y_\beta)\frac{\partial}{\partial y_\sigma}y_\gamma - y_\gamma y_\alpha y_\beta - i\epsilon_{\delta\sigma}\frac{\partial}{\partial y_\delta}y_\gamma\frac{\partial}{\partial y_\sigma}(y_\alpha y_\beta)\\
&= i\epsilon_{\delta\sigma}(\delta^\delta_\alpha y_\beta + y_\alpha\delta^\delta_\beta)\delta^\sigma_\gamma - i\epsilon_{\delta\sigma}\delta^\delta_\gamma(\delta^\sigma_\alpha y_\beta + y_\alpha\delta^\sigma_\beta)\\
&= 2i\epsilon_{\alpha\gamma}y_\beta + 2i\epsilon_{\beta\gamma}y_\alpha = y_\alpha[y_\beta, y_\gamma]_* + [y_\alpha, y_\gamma]_* y_\beta\ .
\end{aligned}
$$

Show

$$
[P_{\mu_1}\cdots P_{\mu_n}, K_\mu]_* = [P_{\mu_1}, K_\mu]_* P_{\mu_2}\cdots P_{\mu_n} + P_{\mu_1}[P_{\mu_2}, K_a]_* P_{\mu_3}\cdots P_{\mu_n} + \cdots + P_{\mu_1}\cdots P_{\mu_{n-1}}[P_{\mu_n}, K_a]_*\ .
$$

Same proof is valid for $[D, P_{\mu_1}\ldots P_{\mu_n}]_* = -nP_{\mu_1}\ldots P_{\mu_n}$.
Each P and K has two spinors (twistors) y_α, thus we have to show

$$
\begin{aligned}
[y_{\alpha_1}y_{\alpha_2}y_{\alpha_3}\cdots y_{\alpha_n}, y_\gamma y_\delta]_* = [y_{\alpha_1}y_{\alpha_2}, y_\gamma y_\delta]y_{\alpha_3}\cdots y_{\alpha_n} + y_{\alpha_1}y_{\alpha_2}[y_{\alpha_3}y_{\alpha_4}, y_\gamma y_\delta]y_{\alpha_5}\cdots y_{\alpha_n} + \cdots\\
\cdots + y_{\alpha_1}\cdots y_{\alpha_{n-2}}[y_{\alpha_{n-1}}y_{\alpha_n}, y_\gamma y_\delta]\ .
\end{aligned}
$$

We start with

$$
\begin{aligned}
[y_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n}, y_\gamma y_\delta]_* &= y_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n}y_\gamma y_\delta + i\epsilon_{\epsilon\varphi}\frac{\partial}{\partial y_\epsilon}(y_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n})\frac{\partial}{\partial y_\varphi}(y_\gamma y_\delta)\\
&\quad - \frac{1}{2}\epsilon_{\pi\sigma}\epsilon_{\epsilon\varphi}\frac{\partial}{\partial y_\pi}\frac{\partial}{\partial y_\epsilon}(y_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n})\frac{\partial}{\partial y_\sigma}\frac{\partial}{\partial y_\varphi}(y_\gamma y_\delta) + \underbrace{\cdots}_{=0}\\
&\quad - y_\gamma y_\delta y_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n} - i\epsilon_{\epsilon\varphi}\frac{\partial}{\partial y_\epsilon}(y_\gamma y_\delta)\frac{\partial}{\partial y_\varphi}(y_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n})\\
&\quad + \frac{1}{2}\epsilon_{\pi\sigma}\epsilon_{\epsilon\varphi}\frac{\partial}{\partial y_\pi}\frac{\partial}{\partial y_\epsilon}(y_\gamma y_\delta)\frac{\partial}{\partial y_\sigma}\frac{\partial}{\partial y_\varphi}(y_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n}) + \underbrace{\cdots}_{=0}\\
&= i\epsilon_{\epsilon\varphi}\Big(\delta^\epsilon_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n} + y_{\alpha_1}\delta^\epsilon_{\alpha_2}y_{\alpha_3}\cdots y_{\alpha_n} + \cdots + y_{\alpha_1}\cdots y_{\alpha_{n-1}}\delta^\epsilon_{\alpha_n}\Big)\Big(\delta^\varphi_\gamma y_\delta + y_\gamma\delta^\varphi_\delta\Big)\\
&\quad - \frac{1}{2}\epsilon_{\pi\sigma}\epsilon_{\epsilon\varphi}\frac{\partial}{\partial y_\pi}\Big(\delta^\epsilon_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n} + y_{\alpha_1}\delta^\epsilon_{\alpha_2}y_{\alpha_3}\cdots y_{\alpha_n} + \cdots + y_{\alpha_1}\cdots y_{\alpha_{n-1}}\delta^\epsilon_{\alpha_n}\Big)\frac{\partial}{\partial y_\sigma}\Big(\delta^\varphi_\gamma y_\delta + y_\gamma\delta^\varphi_\delta\Big)\\
&\quad - i\epsilon_{\epsilon\varphi}\Big(\delta^\epsilon_\gamma y_\delta + y_\gamma\delta^\epsilon_\delta\Big)\Big(\delta^\varphi_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n} + y_{\alpha_1}\delta^\varphi_{\alpha_2}y_{\alpha_3}\cdots y_{\alpha_n} + \cdots + y_{\alpha_1}\cdots y_{\alpha_{n-1}}\delta^\varphi_{\alpha_n}\Big)\\
&\quad + \frac{1}{2}\epsilon_{\pi\sigma}\epsilon_{\epsilon\varphi}\frac{\partial}{\partial y_\pi}\Big(\delta^\epsilon_\gamma y_\delta + y_\gamma\delta^\epsilon_\delta\Big)\frac{\partial}{\partial y_\sigma}\Big(\delta^\varphi_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n} + y_{\alpha_1}\delta^\varphi_{\alpha_2}y_{\alpha_3}\cdots y_{\alpha_n} + \cdots + y_{\alpha_1}\cdots y_{\alpha_{n-1}}\delta^\varphi_{\alpha_n}\Big)\\
&= i\Big(\epsilon_{\alpha_1\gamma}y_\delta y_{\alpha_2}\cdots y_{\alpha_n} + y_{\alpha_1}\epsilon_{\alpha_2\gamma}y_\delta y_{\alpha_3}\cdots y_{\alpha_n} + \cdots + y_{\alpha_1}\cdots y_{\alpha_{n-1}}\epsilon_{\alpha_n\gamma}y_\delta\Big)\\
&\quad + i\Big(\epsilon_{\alpha_1\delta}y_\gamma y_{\alpha_2}\cdots y_{\alpha_n} + y_{\alpha_1}\epsilon_{\alpha_2\delta}y_\gamma y_{\alpha_3}\cdots y_{\alpha_n} + \cdots + y_{\alpha_1}\cdots y_{\alpha_{n-1}}\epsilon_{\alpha_n\delta}y_\gamma\Big)\\
&\quad - \frac{1}{2}\epsilon_{\pi\sigma}\epsilon_{\epsilon\varphi}\frac{\partial}{\partial y_\pi}\Big(\delta^\epsilon_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n} + y_{\alpha_1}\delta^\epsilon_{\alpha_2}y_{\alpha_3}\cdots y_{\alpha_n} + \cdots + y_{\alpha_1}\cdots y_{\alpha_{n-1}}\delta^\epsilon_{\alpha_n}\Big)\Big(\delta^\varphi_\gamma\delta^\sigma_\delta + \delta^\sigma_\gamma\delta^\varphi_\delta\Big)\\
&\quad - i\Big(\epsilon_{\gamma\alpha_1}y_\delta y_{\alpha_2}\cdots y_{\alpha_n} + y_{\alpha_1}\epsilon_{\gamma\alpha_2}y_\delta y_{\alpha_3}\cdots y_{\alpha_n} + \cdots + y_{\alpha_1}\cdots y_{\alpha_{n-1}}\epsilon_{\gamma\alpha_n}y_\delta\Big)\\
&\quad - i\Big(\epsilon_{\delta\alpha_1}y_\gamma y_{\alpha_2}\cdots y_{\alpha_n} + y_{\alpha_1}\epsilon_{\delta\alpha_2}y_\gamma y_{\alpha_3}\cdots y_{\alpha_n} + \cdots + y_{\alpha_1}\cdots y_{\alpha_{n-1}}\epsilon_{\delta\alpha_n}y_\gamma\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\epsilon_{\pi\sigma}\epsilon_{\epsilon\varphi}\left(\delta^{\epsilon}_{\gamma}\delta^{\pi}_{\delta}+\delta^{\pi}_{\gamma}\delta^{\epsilon}_{\delta}\right)\frac{\partial}{\partial y_{\sigma}}\left(\delta^{\varphi}_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n}+y_{\alpha_1}\delta^{\varphi}_{\alpha_2}y_{\alpha_3}\cdots y_{\alpha_n}+\cdots+y_{\alpha_1}\cdots y_{\alpha_{n-1}}\delta^{\varphi}_{\alpha_n}\right)\\
=&\left(2i\epsilon_{\alpha_1\gamma}y_{\alpha_2}y_{\delta}+2i\epsilon_{\alpha_1\delta}y_{\alpha_2}y_{\gamma}+2i\epsilon_{\alpha_2\gamma}y_{\alpha_1}y_{\delta}+2i\epsilon_{\alpha_2\delta}y_{\alpha_1}y_{\gamma}\right)y_{\alpha_3}\cdots y_{\alpha_n}\\
&+y_{\alpha_1}y_{\alpha_2}\left(2i\epsilon_{\alpha_3\gamma}y_{\alpha_4}y_{\delta}+2i\epsilon_{\alpha_3\delta}y_{\alpha_4}y_{\gamma}+2i\epsilon_{\alpha_4\gamma}y_{\alpha_3}y_{\delta}+2i\epsilon_{\alpha_4\delta}y_{\alpha_3}y_{\gamma}\right)y_{\alpha_5}\cdots y_{\alpha_n}+\cdots\\
&+y_{\alpha_1}\cdots y_{\alpha_{n-2}}\left(2i\epsilon_{\alpha_{n-1}\gamma}y_{\alpha_n}y_{\delta}+2i\epsilon_{\alpha_{n-1}\delta}y_{\alpha_m}y_{\gamma}+2i\epsilon_{\alpha_n\gamma}y_{\alpha_{n-1}}y_{\delta}+2i\epsilon_{\alpha_n\delta}y_{\alpha_{n-1}}y_{\gamma}\right)\\
&-\frac{1}{2}\left(\epsilon_{\pi\delta}\epsilon_{\epsilon\gamma}+\epsilon_{\pi\gamma}\epsilon_{\epsilon\delta}\right)\frac{\partial}{\partial y_{\pi}}\left(\delta^{\epsilon}_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n}+y_{\alpha_1}\delta^{\epsilon}_{\alpha_2}y_{\alpha_3}\cdots y_{\alpha_n}+\cdots+y_{\alpha_1}\cdots y_{\alpha_{n-1}}\delta^{\epsilon}_{\alpha_n}\right)\\
&+\frac{1}{2}\left(\epsilon_{\delta\sigma}\epsilon_{\gamma\varphi}+\epsilon_{\gamma\sigma}\epsilon_{\delta\varphi}\right)\frac{\partial}{\partial y_{\sigma}}\left(\delta^{\varphi}_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n}+y_{\alpha_1}\delta^{\varphi}_{\alpha_2}y_{\alpha_3}\cdots y_{\alpha_n}+\cdots+y_{\alpha_1}\cdots y_{\alpha_{n-1}}\delta^{\varphi}_{\alpha_n}\right)
\end{aligned}
$$

and finally setting σ to π and φ to ϵ we get

$$
=[y_{\alpha_1}y_{\alpha_2},y_{\gamma}y_{\delta}]y_{\alpha_3}\cdots y_{\alpha_n}+y_{\alpha_1}y_{\alpha_2}[y_{\alpha_3}y_{\alpha_4},y_{\gamma}y_{\delta}]y_{\alpha_5}\cdots y_{\alpha_n}+\cdots+y_{\alpha_1}\cdots y_{\alpha_{n-2}}[y_{\alpha_{n-1}}y_{\alpha_n},y_{\gamma}y_{\delta}]\ .
$$

Expressed by the two-component spinors (twistors) the dilatation operator gets

$$
\begin{aligned}
D&=\frac{1}{2}\epsilon_{\beta\alpha}y^{+\alpha}y^{-\beta}=\frac{1}{8}\epsilon_{\beta\alpha}(y^{\alpha}-i\bar{y}^{\alpha})(\bar{y}^{\beta}-iy^{\beta})=\frac{1}{8}\epsilon_{\beta\alpha}(y^{\alpha}\bar{y}^{\beta}-iy^{\alpha}y^{\beta}-i\bar{y}^{\alpha}\bar{y}^{\beta}-\bar{y}^{\alpha}y^{\beta})\\
&=\frac{1}{8}\epsilon_{\beta\alpha}[(y^{+\alpha}+iy^{-\alpha})(y^{-\beta}+iy^{+\beta})+i(y^{+\alpha}+iy^{-\alpha})(y^{+\beta}+iy^{-\beta})\\
&\quad+i(y^{-\alpha}+iy^{+\alpha})(y^{-\beta}+iy^{+\beta})-(y^{-\alpha}+iy^{+\alpha})(y^{+\beta}+iy^{-\beta})]\\
&=\frac{1}{8}\epsilon_{\beta\alpha}[y^{+\alpha}y^{-\beta}+iy^{-\alpha}y^{-\beta}+iy^{+\alpha}y^{+\beta}-y^{-\alpha}y^{-\beta}+iy^{+\alpha}y^{+\beta}-y^{-\alpha}y^{+\beta}-y^{+\alpha}y^{-\beta}-iy^{-\alpha}y^{-\beta}\\
&\quad-iy^{-\alpha}y^{-\beta}+y^{-\alpha}y^{+\beta}-y^{+\alpha}y^{-\beta}+iy^{+\alpha}y^{+\beta}-y^{-\alpha}y^{+\beta}-iy^{-\alpha}y^{-\beta}-iy^{+\alpha}y^{+\beta}+y^{+\alpha}y^{-\beta}]\\
&=\frac{1}{8}\epsilon_{\beta\alpha}(2iy^{+\alpha}y^{+\beta}-2iy^{-\alpha}y^{-\beta}+2y^{+\alpha}y^{-\beta}-2y^{-\alpha}y^{+\beta})=\frac{1}{4}\epsilon_{\beta\alpha}(y^{+\alpha}+iy^{-\alpha})(y^{-\beta}+iy^{+\beta})\\
&=\frac{1}{4}\epsilon_{\beta\alpha}y^{\alpha}\bar{y}^{\beta}\ .
\end{aligned}
$$

C.2 Gravity-side commutators

In the following we calculate the nine commutation relations of the conformal algebra of the four generators D, $P_{\alpha\beta}$, $L_{\alpha\beta}$ and $K_{\alpha\beta}$ of chapter 3.2. To be able to use the introduced star product, the generators must be rewritten with the two-component spinors (twistors) y_α and $\bar{y}_\alpha$ instead of the oscillators y^+_α and y^-_α, as

$$\begin{aligned}
D &= \frac{1}{2}y^{+\alpha}y^-_\alpha = -\frac{1}{4}y_\alpha\bar{y}^\alpha = -\frac{1}{4}\epsilon^{\alpha\beta}y_\alpha\bar{y}_\beta \ , \\
P_{\alpha\beta} &= iy^-_\alpha y^-_\beta = \frac{i}{4}\Big(\bar{y}_\alpha\bar{y}_\beta - iy_\alpha\bar{y}_\beta - i\bar{y}_\alpha y_\beta - y_\alpha y_\beta\Big) = -K_{\alpha\beta}(y \leftrightarrow \bar{y}) \ , \\
L_{\alpha\beta} &= \epsilon_{\delta\alpha}L^\delta{}_\beta = \epsilon_{\delta\alpha}\Big(y^{+\delta}y^-_\beta - \frac{1}{2}\delta^\delta_\beta y^{+\gamma}y^-_\gamma\Big) = y^+_\alpha y^-_\beta - \frac{1}{4}\epsilon_{\delta\alpha}\delta^\delta_\beta y^\gamma\bar{y}_\gamma \\
&= \frac{1}{4}(y_\alpha - i\bar{y}_\alpha)(\bar{y}_\beta - iy_\beta) - \frac{1}{4}\epsilon_{\delta\alpha}\delta^\delta_\beta\epsilon^{\gamma\epsilon}y_\epsilon\bar{y}_\gamma \\
&= \frac{1}{4}(y_\alpha\bar{y}_\beta - i\bar{y}_\alpha\bar{y}_\beta - iy_\alpha y_\beta - \bar{y}_\alpha y_\beta) - \frac{1}{4}(\delta^\gamma_\beta\delta^\epsilon_\alpha - \delta^\epsilon_\beta\delta^\gamma_\alpha)y_\epsilon\bar{y}_\gamma = -\frac{i}{4}(\bar{y}_\alpha\bar{y}_\beta + y_\alpha y_\beta) \ , \\
K_{\alpha\beta} &= -iy^+_\alpha y^+_\beta = -\frac{i}{4}\Big(y_\alpha y_\beta - i\bar{y}_\alpha y_\beta - iy_\alpha\bar{y}_\beta - \bar{y}_\alpha\bar{y}_\beta\Big) \ .
\end{aligned}$$

For the indices we use the following conventions:
$\alpha,\beta,\gamma,\delta\ldots = 1,2$; $\dot{\alpha},\dot{\beta},\dot{\gamma},\dot{\delta}\ldots = 1,2$ (spinor index); e.g. y_α , $y_{\dot{\alpha}}$, $\epsilon^{\alpha\beta}$, $\epsilon^{\dot{\alpha}\dot{\beta}}$
$m,n,\ldots = 0,1,2,3$ (4D-spacetime, dotted/undotted spinor indices); e.g. $(\sigma_m)_{\alpha\dot{\beta}}$, η_{mn}
$\mu,\nu,\ldots = 0,1,3$ (3D, undotted spinor indices, without spacetime index $m=2$); e.g. $(\gamma_\mu)_{\alpha\beta}$, $\eta_{\mu\nu}$

At start all conformal generators must be with 3D-indices and undotted spinor indices.

1. Commutator $[D, P_\mu]_*$

$$\begin{aligned}
[D,P_\mu]_* &= (\gamma_\mu)^{\gamma\delta}[D,P_{\gamma\delta}]_* = -\frac{i}{16}(\gamma_\mu)^{\gamma\delta}\epsilon^{\alpha\beta}\Big[y_\alpha\bar{y}_\beta, \bar{y}_\gamma\bar{y}_\delta - iy_\gamma\bar{y}_\delta - i\bar{y}_\gamma y_\delta - y_\gamma y_\delta\Big]_* \\
&= -\frac{i}{8}(\gamma_\mu)^{\gamma\delta}\epsilon^{\alpha\beta}\Big(i\epsilon_{\beta\gamma}y_\alpha\bar{y}_\delta + i\epsilon_{\beta\delta}y_\alpha\bar{y}_\gamma + \epsilon_{\alpha\gamma}\bar{y}_\beta\bar{y}_\delta + \epsilon_{\beta\delta}y_\alpha y_\gamma \\
&\qquad + \epsilon_{\alpha\delta}\bar{y}_\beta\bar{y}_\gamma + \epsilon_{\beta\gamma}y_\alpha y_\delta - i\epsilon_{\alpha\gamma}\bar{y}_\beta y_\delta - i\epsilon_{\alpha\delta}\bar{y}_\beta y_\gamma\Big) \\
&= \frac{i}{4}(\gamma_\mu)^{\gamma\delta}\Big(y_\gamma y_\delta - \bar{y}_\gamma\bar{y}_\delta + i\bar{y}_\gamma y_\delta + iy_\gamma\bar{y}_\delta\Big) = -(\gamma_\mu)^{\gamma\delta}P_{\gamma\delta} = -P_\mu
\end{aligned}$$

2. Commutator $[D, K_\mu]_*$

$$\begin{aligned}
[D,K_\mu]_* &= (\gamma_\mu)^{\gamma\delta}[D,K_{\gamma\delta}]_* = \frac{i}{16}(\gamma_\mu)^{\gamma\delta}\epsilon^{\alpha\beta}\Big[y_\alpha\bar{y}_\beta, y_\gamma y_\delta - i\bar{y}_\gamma y_\delta - iy_\gamma\bar{y}_\delta - \bar{y}_\gamma\bar{y}_\delta\Big]_* \\
&= \frac{i}{8}(\gamma_\mu)^{\gamma\delta}\epsilon^{\alpha\beta}\Big(i\epsilon_{\alpha\gamma}\bar{y}_\beta y_\delta + i\epsilon_{\alpha\delta}\bar{y}_\beta y_\gamma + \epsilon_{\alpha\delta}\bar{y}_\beta\bar{y}_\gamma + \epsilon_{\beta\gamma}y_\alpha y_\delta \\
&\qquad + \epsilon_{\alpha\gamma}\bar{y}_\beta\bar{y}_\delta + \epsilon_{\beta\delta}y_\alpha y_\gamma - i\epsilon_{\beta\gamma}y_\alpha\bar{y}_\delta - i\epsilon_{\beta\delta}y_\alpha\bar{y}_\gamma\Big) \\
&= \frac{i}{4}(\gamma_\mu)^{\gamma\delta}\Big(- y_\gamma y_\delta + \bar{y}_\gamma\bar{y}_\delta + i\bar{y}_\gamma y_\delta + iy_\gamma\bar{y}_\delta\Big) = (\gamma_\mu)^{\gamma\delta}K_{\gamma\delta} = K_\mu
\end{aligned}$$

3. Commutator $[K_\mu, P_\nu]_*$

$$[K_\mu, P_\nu]_* = (\gamma_\mu)^{\alpha\beta}(\gamma_\nu)^{\gamma\delta}[K_{\alpha\beta}, P_{\gamma\delta}]_*$$
$$= \frac{1}{16}(\gamma_\mu)^{\alpha\beta}(\gamma_\nu)^{\gamma\delta}\Big[y_\alpha y_\beta - i\bar{y}_\alpha y_\beta - iy_\alpha\bar{y}_\beta - \bar{y}_\alpha\bar{y}_\beta, \bar{y}_\gamma\bar{y}_\delta - iy_\gamma\bar{y}_\delta - i\bar{y}_\gamma y_\delta - y_\gamma y_\delta\Big]_*$$

Now we move from the undotted 3D spinors with Gamma matrices (γ_μ) to the dotted/undotted 4D spinors with Sigma matrices (σ_m) and continue with

$$\begin{aligned}
[K_\mu, P_\nu]_* \mapsto \frac{1}{8}\Big[&(\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\dot{\delta}}\Big(\epsilon_{\alpha\gamma}y_\beta\bar{y}_{\dot{\delta}} + \epsilon_{\beta\gamma}y_\alpha\bar{y}_{\dot{\delta}}\Big) + (\sigma_m)^{\alpha\beta}(\sigma_n)^{\dot{\gamma}\delta}\Big(\epsilon_{\alpha\delta}y_\beta\bar{y}_{\dot{\gamma}} + \epsilon_{\beta\delta}y_\alpha\bar{y}_{\dot{\gamma}}\Big)\\
&- i(\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\delta}\Big(\epsilon_{\alpha\gamma}y_\beta y_\delta + \epsilon_{\alpha\delta}y_\beta y_\gamma + \epsilon_{\beta\gamma}y_\alpha y_\delta + \epsilon_{\beta\delta}y_\alpha y_\gamma\Big) + (\sigma_m)^{\dot{\alpha}\beta}(\sigma_n)^{\dot{\gamma}\dot{\delta}}\Big(\epsilon_{\dot{\alpha}\dot{\gamma}}y_\beta\bar{y}_{\dot{\delta}} + \epsilon_{\dot{\alpha}\dot{\delta}}y_\beta\bar{y}_{\dot{\gamma}}\Big)\\
&- i(\sigma_m)^{\dot{\alpha}\beta}(\sigma_n)^{\gamma\dot{\delta}}\Big(\epsilon_{\dot{\alpha}\dot{\delta}}y_\beta y_\gamma + \epsilon_{\beta\gamma}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\delta}}\Big) - i(\sigma_m)^{\dot{\alpha}\beta}(\sigma_n)^{\dot{\gamma}\delta}\Big(\epsilon_{\dot{\alpha}\dot{\gamma}}y_\beta y_\delta + \epsilon_{\beta\delta}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\gamma}}\Big)\\
&- (\sigma_m)^{\dot{\alpha}\beta}(\sigma_n)^{\gamma\delta}\Big(\epsilon_{\beta\gamma}\bar{y}_{\dot{\alpha}}y_\delta + \epsilon_{\beta\delta}\bar{y}_{\dot{\alpha}}y_\gamma\Big) + (\sigma_m)^{\alpha\dot{\beta}}(\sigma_n)^{\dot{\gamma}\dot{\delta}}\Big(\epsilon_{\dot{\beta}\dot{\gamma}}y_\alpha\bar{y}_{\dot{\delta}} + \epsilon_{\dot{\beta}\dot{\delta}}y_\alpha\bar{y}_{\dot{\gamma}}\Big)\\
&- i(\sigma_m)^{\alpha\dot{\beta}}(\sigma_n)^{\gamma\dot{\delta}}\Big(\epsilon_{\dot{\beta}\dot{\delta}}y_\alpha y_\gamma + \epsilon_{\alpha\gamma}\bar{y}_{\dot{\beta}}\bar{y}_{\dot{\delta}}\Big) - i(\sigma_m)^{\alpha\dot{\beta}}(\sigma_n)^{\dot{\gamma}\delta}\Big(\epsilon_{\dot{\beta}\dot{\gamma}}y_\alpha y_\delta + \epsilon_{\alpha\delta}\bar{y}_{\dot{\beta}}\bar{y}_{\dot{\gamma}}\Big)\\
&- (\sigma_m)^{\alpha\dot{\beta}}(\sigma_n)^{\gamma\delta}\Big(\epsilon_{\alpha\gamma}\bar{y}_{\dot{\beta}}y_\delta + \epsilon_{\alpha\delta}\bar{y}_{\dot{\beta}}y_\gamma\Big) - i(\sigma_m)^{\dot{\alpha}\dot{\beta}}(\sigma_n)^{\dot{\gamma}\dot{\delta}}\Big(\epsilon_{\dot{\alpha}\dot{\gamma}}\bar{y}_{\dot{\beta}}\bar{y}_{\dot{\delta}} + \epsilon_{\dot{\alpha}\dot{\delta}}\bar{y}_{\dot{\beta}}\bar{y}_{\dot{\gamma}} + \epsilon_{\dot{\beta}\dot{\gamma}}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\delta}} + \epsilon_{\dot{\beta}\dot{\delta}}\bar{y}_{\dot{\alpha}}y_{\dot{\gamma}}\Big)\\
&- (\sigma_m)^{\dot{\alpha}\dot{\beta}}(\sigma_n)^{\gamma\dot{\delta}}\Big(\epsilon_{\dot{\alpha}\dot{\delta}}\bar{y}_{\dot{\beta}}y_\gamma + \epsilon_{\dot{\beta}\dot{\delta}}\bar{y}_{\dot{\alpha}}y_\gamma\Big) - (\sigma_m)^{\dot{\alpha}\dot{\beta}}(\sigma_n)^{\dot{\gamma}\delta}\Big(\epsilon_{\dot{\alpha}\dot{\gamma}}\bar{y}_{\dot{\beta}}y_\delta + \epsilon_{\dot{\beta}\dot{\gamma}}\bar{y}_{\dot{\alpha}}y_\delta\Big)\Big]
\end{aligned} \tag{C.1}$$

Now we perform the following auxiliary calculation:

$$(\sigma_m)_{\alpha\dot{\alpha}}(\sigma_n)^{\beta\dot{\alpha}} + (\sigma_n)_{\alpha\dot{\alpha}}(\sigma_m)^{\beta\dot{\alpha}} = 2\delta_\alpha^\beta\eta_{mn}$$

and

$$(\sigma_{mn})_{\alpha\beta} = \frac{1}{4}\Big((\sigma_m)_{\alpha\dot{\alpha}}(\sigma_n)_\beta{}^{\dot{\alpha}} - (\sigma_n)_{\alpha\dot{\alpha}}(\sigma_m)_\beta{}^{\dot{\alpha}}\Big)$$

leads to

$$(\sigma_{mn})_\alpha{}^\beta = \frac{1}{4}\Big((\sigma_m)_{\alpha\dot{\alpha}}(\sigma_n)^{\beta\dot{\alpha}} - (\sigma_n)_{\alpha\dot{\alpha}}(\sigma_m)^{\beta\dot{\alpha}}\Big) = \frac{1}{4}\Big(2\delta_\alpha^\beta\eta_{mn} - (\sigma_n)_{\alpha\dot{\alpha}}(\sigma_m)^{\beta\dot{\alpha}} - (\sigma_n)_{\alpha\dot{\alpha}}(\sigma_m)^{\beta\dot{\alpha}}\Big)$$
$$= \frac{1}{2}\Big(\delta_\alpha^\beta\eta_{mn} - (\sigma_n)_{\alpha\dot{\alpha}}(\sigma_m)^{\beta\dot{\alpha}}\Big)$$

and we receive

$$(\sigma_m)_{\alpha\dot{\alpha}}(\sigma_n)^{\beta\dot{\alpha}} = \eta_{mn}\delta_\alpha^\beta + 2(\sigma_{mn})_\alpha{}^\beta \text{ as well as } (\sigma_m)_{\alpha\dot{\alpha}}(\sigma_n)^{\alpha\dot{\beta}} = \eta_{mn}\delta_{\dot{\alpha}}^{\dot{\beta}} + 2(\bar{\sigma}_{mn})_{\dot{\alpha}}{}^{\dot{\beta}}\ . \tag{C.2}$$

Now we analyze all terms of (C.1). We use $\delta^{\alpha\dot\beta} = -\delta^{\dot\beta\alpha}$, $\delta^{\alpha\dot\beta} y_{\dot\beta} = y^\alpha$ and $\delta^{\dot\alpha\beta} y_{\dot\alpha} = -y^\beta$.
For the unbarred/barred $(y\bar y)$-terms, we get:

$$(\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\dot\delta}\epsilon_{\alpha\gamma}y_\beta\bar y_{\dot\delta} = (\sigma_m)_\gamma{}^\beta(\sigma_n)^{\gamma\dot\delta}y_\beta\bar y_{\dot\delta} = \eta_{mn}\delta^{\beta\dot\delta}y_\beta\bar y_{\dot\delta} + 2\underbrace{(\bar\sigma_{mn})^{\beta\dot\delta}}_{=0}y_\beta\bar y_{\dot\delta} = \eta_{mn}y_\beta\bar y^\beta = -\eta_{mn}y_\alpha\bar y^\alpha$$

$$(\sigma_m)^{\alpha\dot\beta}(\sigma_n)^{\dot\gamma\dot\delta}\epsilon_{\dot\beta\dot\delta}y_\alpha\bar y_{\dot\gamma} = (\sigma_m)^\alpha{}_{\dot\delta}(\sigma_n)^{\dot\gamma\dot\delta}y_\alpha\bar y_{\dot\gamma} = \eta_{mn}\delta^{\alpha\dot\gamma}y_\alpha\bar y_{\dot\gamma} + 2\underbrace{(\sigma_{mn})^{\alpha\dot\gamma}}_{=0}y_\alpha\bar y_{\dot\gamma} = -\eta_{mn}y_\alpha\bar y^\alpha$$

$(\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\dot\delta}\epsilon_{\beta\gamma}y_\alpha\bar y_{\dot\delta} = (\sigma_m)^\alpha{}_\gamma(\sigma_n)^{\gamma\dot\delta}y_\alpha\bar y_{\dot\delta} = 0$ (because index structure does not correspond to (C.2))

$$(\sigma_m)^{\alpha\beta}(\sigma_n)^{\dot\gamma\delta}\epsilon_{\alpha\delta}y_\beta\bar y_{\dot\gamma} = 0\ ,\ (\sigma_m)^{\alpha\beta}(\sigma_n)^{\dot\gamma\delta}\epsilon_{\beta\delta}y_\alpha\bar y_{\dot\gamma} = 0\ ,\ (\sigma_m)^{\alpha\dot\beta}(\sigma_n)^{\dot\gamma\dot\delta}\epsilon_{\dot\beta\dot\gamma}y_\alpha\bar y_{\dot\delta} = 0$$

$$(\sigma_m)^{\dot\alpha\beta}(\sigma_n)^{\dot\gamma\dot\delta}\epsilon_{\dot\alpha\dot\gamma}y_\beta\bar y_{\dot\delta} = 0\ ,\ (\sigma_m)^{\dot\alpha\beta}(\sigma_n)^{\dot\gamma\dot\delta}\epsilon_{\dot\alpha\dot\delta}y_\beta\bar y_{\dot\gamma} = 0$$

For the unbarred (yy)-terms, we get:

$$(\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\delta}\epsilon_{\alpha\gamma}y_\beta y_\delta = (\sigma_m)_\gamma{}^\beta(\sigma_n)^{\gamma\delta}y_\beta y_\delta = \eta_{mn}\underbrace{\delta^{\beta\delta}y_\beta y_\delta}_{=0} + 2(\sigma_{mn})^{\beta\delta}y_\beta y_\delta = 2(\sigma_{mn})^{\alpha\beta}y_\alpha y_\beta$$

$$(\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\delta}\epsilon_{\beta\delta}y_\alpha y_\gamma = (\sigma_m)^\alpha{}_\delta(\sigma_n)^{\gamma\delta}y_\alpha y_\gamma = \eta_{mn}\delta^{\alpha\gamma}y_\alpha y_\gamma + 2(\sigma_{mn})^{\alpha\gamma}y_\alpha y_\gamma = 2(\sigma_{mn})^{\alpha\beta}y_\alpha y_\beta$$

$$(\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\delta}\epsilon_{\beta\gamma}y_\alpha y_\delta = 0\ ,\ (\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\delta}\epsilon_{\alpha\delta}y_\beta y_\gamma = 0\ ,\ (\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\delta}\epsilon_{\beta\delta}y_\alpha y_\gamma = 0$$

$$(\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\delta}\epsilon_{\beta\gamma}y_\alpha y_\delta = 0\ ,\ (\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\delta}\epsilon_{\alpha\gamma}y_\beta y_\delta = 0\ ,\ (\sigma_m)^{\alpha\beta}(\sigma_n)^{\gamma\delta}\epsilon_{\alpha\delta}y_\beta y_\gamma = 0$$

For the barred-unbarred $(\bar y y)$-terms, we get:

$$(\sigma_m)^{\alpha\dot\beta}(\sigma_n)^{\gamma\delta}\epsilon_{\alpha\gamma}\bar y_{\dot\beta}y_\delta = (\sigma_m)_\gamma{}^{\dot\beta}(\sigma_n)^{\gamma\delta}\bar y_{\dot\beta}y_\delta = \eta_{mn}\delta^{\dot\beta\delta}\bar y_{\dot\beta}y_\delta + 2\underbrace{(\bar\sigma_{mn})^{\dot\beta\delta}}_{=0}\bar y_{\dot\beta}y_\delta = -\eta_{mn}\bar y^\delta y_\delta = \eta_{mn}y_\alpha\bar y^\alpha$$

$$(\sigma_m)^{\dot\alpha\dot\beta}(\sigma_n)^{\gamma\dot\delta}\epsilon_{\dot\beta\dot\delta}\bar y_{\dot\alpha}y_\gamma = (\sigma_m)^{\dot\alpha}{}_{\dot\delta}(\sigma_n)^{\gamma\dot\delta}\bar y_{\dot\alpha}y_\gamma = \eta_{mn}\delta^{\dot\alpha\gamma}\bar y_{\dot\alpha}y_\gamma + 2\underbrace{(\sigma_{mn})^{\dot\alpha\gamma}}_{=0}\bar y_{\dot\alpha}\bar y_{\dot\gamma} = \eta_{mn}y_\alpha\bar y^\alpha$$

and as above, the remaining terms are 0

For the barred $(\bar y\bar y)$-terms, we get:

$$(\sigma_m)^{\alpha\dot\beta}(\sigma_n)^{\gamma\dot\delta}\epsilon_{\alpha\gamma}\bar y_{\dot\beta}\bar y_{\dot\delta} = (\sigma_m)_\gamma{}^{\dot\beta}(\sigma_n)^{\gamma\dot\delta}\bar y_{\dot\beta}\bar y_{\dot\delta} = \eta_{mn}\underbrace{\delta^{\dot\beta\dot\delta}\bar y_{\dot\beta}\bar y_{\dot\delta}}_{=0} + 2(\bar\sigma_{mn})^{\dot\beta\dot\delta}\bar y_{\dot\beta}\bar y_{\dot\delta} = 2(\bar\sigma_{mn})^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\bar y_{\dot\beta}$$

$$(\sigma_m)^{\dot\alpha\beta}(\sigma_n)^{\dot\gamma\delta}\epsilon_{\beta\delta}\bar y_{\dot\alpha}\bar y_{\dot\gamma} = (\sigma_m)^{\dot\alpha}{}_\delta(\sigma_n)^{\dot\gamma\delta}\bar y_{\dot\alpha}\bar y_{\dot\gamma} = \eta_{mn}\delta^{\dot\alpha\dot\gamma}\bar y_{\dot\alpha}\bar y_{\dot\gamma} + 2(\bar\sigma_{mn})^{\dot\alpha\dot\gamma}\bar y_{\dot\alpha}\bar y_{\dot\gamma} = 2(\bar\sigma_{mn})^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\bar y_{\dot\beta}$$

and as above, the remaining terms are 0 .

Using all this we continue with

$$[K_m, P_n]_* \mapsto \frac{1}{8}\Big(-2\eta_{mn}y_\alpha\bar y^\alpha - 4i(\sigma_{mn})^{\alpha\beta}y_\alpha y_\beta - 2\eta_{mn}y_\alpha\bar y^\alpha - 4i(\bar\sigma_{mn})^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\bar y_{\dot\beta}\Big)$$
$$= -\frac{1}{2}\eta_{mn}y_\alpha\bar y^\alpha - \frac{i}{2}\Big((\sigma_{mn})^{\alpha\beta}y_\alpha y_\beta + (\bar\sigma_{mn})^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\bar y_{\dot\beta})\Big)$$

and we finally back-transform from dotted/undotted 4D spinors with (σ_m) to the undotted 3D spinors with (γ_μ) to finish with

$$[K_\mu, P_\nu]_* = -\frac{1}{2}\eta_{\mu\nu}y_\alpha\bar y^\alpha - \frac{i}{2}(\gamma_{\mu\nu})^{\alpha\beta}(y_\alpha y_\beta + \bar y_\alpha\bar y_\beta) = 2\eta_{\mu\nu}D + 2(\gamma_{\mu\nu})^{\alpha\beta}L_{\alpha\beta} = \underline{2\eta_{\mu\nu}D + 2L_{\mu\nu}}$$

4. Commutator $[L_{\mu\nu}, L_{\rho\sigma}]_*$

$$[L_{\mu\nu}, L_{\rho\sigma}]_* = (\gamma_{\mu\nu})^{\alpha\beta}(\gamma_{\rho\sigma})^{\gamma\delta}[L_{\alpha\beta}, L_{\gamma\delta}]_* = -\frac{1}{16}(\gamma_{\mu\nu})^{\alpha\beta}(\gamma_{\rho\sigma})^{\gamma\delta}\Big[\bar{y}_\alpha\bar{y}_\beta + y_\alpha y_\beta, \bar{y}_\gamma\bar{y}_\delta + y_\gamma y_\delta\Big]_*$$

Move from the undotted 3D spinors with (γ_μ) to the dotted/undotted 4D spinors with (σ_m).

$$\begin{aligned}[L_{\mu\nu}, L_{\rho\sigma}]_* \mapsto -\frac{i}{8}\Big[&(\sigma_{mn})^{\alpha\beta}(\sigma_{rs})^{\gamma\delta}\Big(\epsilon_{\alpha\gamma}y_\beta y_\delta + \epsilon_{\alpha\delta}y_\beta y_\gamma + \epsilon_{\beta\gamma}y_\alpha y_\delta + \epsilon_{\beta\delta}y_\alpha y_\gamma\Big)\\
&+ (\bar{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}}(\bar{\sigma}_{rs})^{\dot{\gamma}\dot{\delta}}\Big(\epsilon_{\dot{\alpha}\dot{\gamma}}\bar{y}_{\dot{\beta}}\bar{y}_{\dot{\delta}} + \epsilon_{\dot{\alpha}\dot{\delta}}\bar{y}_{\dot{\beta}}\bar{y}_{\dot{\gamma}} + \epsilon_{\dot{\beta}\dot{\gamma}}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\delta}} + \epsilon_{\dot{\beta}\dot{\delta}}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\gamma}}\Big)\Big]\end{aligned}$$

Now we need the following auxiliary calculation:

$$\begin{aligned}(\sigma_{mn})^{\alpha\beta}(\sigma_{rs})^{\gamma\delta}\epsilon_{\alpha\gamma} &= (\sigma_{mn})_\gamma{}^\beta(\sigma_{rs})^{\gamma\delta} = \frac{1}{4}\Big((\sigma_m)_{\gamma\dot{\alpha}}(\sigma_n)^{\beta\dot{\alpha}} - (\sigma_n)_{\gamma\dot{\alpha}}(\sigma_m)^{\beta\dot{\alpha}}\Big)\frac{1}{4}\Big((\sigma_r)^\gamma{}_{\dot{\alpha}}(\sigma_s)^{\delta\dot{\alpha}} - (\sigma_s)^\gamma{}_{\dot{\alpha}}(\sigma_r)^{\delta\dot{\alpha}}\Big)\\
&= \frac{1}{16}\Big(2\delta_{mn}\delta^\beta_\gamma\delta^{\dot{\alpha}}_{\dot{\alpha}} - 2\delta_{nm}\delta^\beta_\gamma\delta^{\dot{\alpha}}_{\dot{\alpha}}\Big)\Big(2\delta_{rs}\delta^{\gamma\delta}\delta^{\dot{\alpha}}_{\dot{\alpha}} - 2\delta_{sr}\delta^{\gamma\delta}\delta^{\dot{\alpha}}_{\dot{\alpha}}\Big) = 4\delta_{mn}\delta^\beta_\gamma\delta_{rs}\delta^{\gamma\delta} = 4\delta_{mn}\delta_{rs}\delta^{\beta\delta}\\
&= 4\delta_{tn}\delta^t_m\delta_{rs}\delta^{\beta\delta} = 4\delta_{tn}\eta_{mr}\eta^{rt}\delta_{rs}\delta^{\beta\delta} = 4\eta_{mr}\eta^r{}_n\delta_{rs}\delta^{\beta\delta} = -4\delta^r_n\eta_{mr}\delta_{rs}\delta^{\beta\delta} = -4\eta_{mr}\delta_{ns}\delta^{\beta\delta}\\
&= -2\eta_{mr}\frac{1}{4}\delta_{ns}8\delta^{\beta\delta} = 2\eta_{mr}\frac{1}{4}\Big(2\delta_{ns}\delta^{\beta\delta}\delta^{\dot{\alpha}}_{\dot{\alpha}} - 2\delta_{sn}\delta^{\beta\delta}\delta^{\dot{\alpha}}_{\dot{\alpha}}\Big)\\
&= -2\eta_{mr}\frac{1}{4}\Big((\sigma_n)^\beta{}_{\dot{\alpha}}(\sigma_s)^{\delta\dot{\alpha}} - (\sigma_s)^\beta{}_{\dot{\alpha}}(\sigma_n)^{\delta\dot{\alpha}}\Big) = -2\eta_{mr}(\sigma_{ns})^{\beta\delta}\end{aligned}$$

as well as

$$(\sigma_{mn})^{\alpha\beta}(\sigma_{rs})^{\gamma\delta}\epsilon_{\alpha\delta} = (\sigma_{mn})_\delta{}^\beta(\sigma_{rs})^{\gamma\delta} = 2\eta_{ms}(\sigma_{nr})^{\beta\gamma} \ .$$

With this we get

$$\begin{aligned}[L_{\mu\nu}, L_{\rho\sigma}]_* \mapsto -\frac{i}{4}\Big(&\eta_{nr}(\sigma_{ms})^{\alpha\delta}y_\alpha y_\delta + \eta_{ms}(\sigma_{nr})^{\beta\gamma}y_\beta y_\gamma - \eta_{mr}(\sigma_{ns})^{\beta\delta}y_\beta y_\delta - \eta_{ns}(\sigma_{mr})^{\alpha\gamma}y_\alpha y_\gamma\\
&+ \eta_{nr}(\bar{\sigma}_{ms})^{\dot{\alpha}\dot{\delta}}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\delta}} + \eta_{mn}(\bar{\sigma}_{nr})^{\dot{\beta}\dot{\gamma}}\bar{y}_{\dot{\beta}}\bar{y}_{\dot{\gamma}} - \eta_{mr}(\bar{\sigma}_{ns})^{\dot{\beta}\dot{\delta}}\bar{y}_{\dot{\beta}}\bar{y}_{\dot{\delta}} - \eta_{ns}(\bar{\sigma}_{mr})^{\dot{\alpha}\dot{\gamma}}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\gamma}}\Big) \ .\end{aligned}$$

Move from the dotted/undotted 4D spinors with (σ_m) to the undotted 3D spinors with (γ_μ).

$$\begin{aligned}[L_{\mu\nu}, L_{\rho\sigma}]_* &= \eta_{\nu\rho}(\gamma_{\mu\sigma})^{\alpha\delta}L_{\alpha\delta} + \eta_{\mu\sigma}(\gamma_{\nu\rho})^{\beta\gamma}L_{\beta\gamma} - \eta_{\mu\rho}(\gamma_{\nu\sigma})^{\beta\delta}L_{\beta\delta} - \eta_{\nu\sigma}(\gamma_{\mu\rho})^{\alpha\gamma}L_{\alpha\gamma}\\
&= \eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\rho}L_{\mu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma}\end{aligned}$$

5. Commutator $[L_{\nu\rho}, K_\mu]_*$

$$[L_{\nu\rho}, K_\mu]_* = (\gamma_{\nu\rho})^{\alpha\beta}(\gamma_\mu)^{\gamma\delta}[L_{\alpha\beta}, K_{\gamma\delta}]_* = -\frac{1}{16}(\gamma_{\nu\rho})^{\alpha\beta}(\gamma_\mu)^{\gamma\delta}\Big[\bar{y}_\alpha\bar{y}_\beta + y_\alpha y_\beta, y_\gamma y_\delta - i\bar{y}_\gamma y_\delta - i y_\gamma \bar{y}_\delta - \bar{y}_\gamma\bar{y}_\delta\Big]_*$$

Move from the undotted 3D spinors with (γ_μ) to the dotted/undotted 4D spinors with (σ_m).

$$\begin{aligned}
[L_{\nu\rho}, K_\mu]_* \mapsto -\frac{1}{16}\Big[&(\sigma_{nr})^{\dot\alpha\dot\beta}\Big((\sigma_m)^{\dot\gamma\delta}(2\epsilon_{\dot\alpha\dot\gamma}\bar{y}_{\dot\beta}y_\delta + 2\epsilon_{\dot\beta\dot\gamma}\bar{y}_{\dot\alpha}y_\delta) + (\sigma_m)^{\gamma\dot\delta}(2\epsilon_{\dot\alpha\dot\delta}\bar{y}_{\dot\beta}y_\gamma + 2\epsilon_{\dot\beta\dot\delta}\bar{y}_{\dot\alpha}y_\gamma)\\
&+ (\sigma_m)^{\dot\gamma\dot\delta}(-2i\epsilon_{\dot\alpha\dot\gamma}\bar{y}_{\dot\beta}\bar{y}_{\dot\delta} - 2i\epsilon_{\dot\beta\dot\gamma}\bar{y}_{\dot\alpha}\bar{y}_{\dot\delta} - 2i\epsilon_{\dot\alpha\dot\delta}\bar{y}_{\dot\beta}\bar{y}_{\dot\gamma} - 2i\epsilon_{\dot\beta\dot\delta}\bar{y}_{\dot\alpha}\bar{y}_{\dot\gamma})\Big)\\
&+ (\sigma_{nr})^{\alpha\beta}\Big((\sigma_m)^{\gamma\delta}(2i\epsilon_{\alpha\gamma}y_\beta y_\delta + 2i\epsilon_{\beta\gamma}y_\alpha y_\delta + 2i\epsilon_{\alpha\delta}y_\beta y_\gamma + 2i\epsilon_{\beta\delta}y_\alpha y_\gamma)\\
&+ (\sigma_m)^{\gamma\dot\delta}(2\epsilon_{\alpha\gamma}y_\beta\bar{y}_{\dot\delta} + 2\epsilon_{\beta\gamma}y_\alpha\bar{y}_{\dot\delta}) + (\sigma_m)^{\dot\gamma\delta}(2\epsilon_{\alpha\delta}y_\beta\bar{y}_{\dot\gamma} + 2\epsilon_{\beta\delta}y_\alpha\bar{y}_{\dot\gamma})\Big)\Big]
\end{aligned}$$

Using $\epsilon_{\alpha\gamma} = \delta_{\alpha\beta}\epsilon^\beta{}_\gamma = -\delta_{\alpha\beta}\delta^\beta_\gamma$.

$$\begin{aligned}
[L_{\nu\rho}, K_\mu]_* \mapsto -\frac{1}{8}\Big[&(\sigma_{nr})^{\dot\alpha\dot\beta}\Big(\delta_{\dot\alpha\dot\beta}(\sigma_m)^{\dot\gamma\delta}(-\delta^{\dot\beta}_{\dot\gamma}\bar{y}_{\dot\beta}y_\delta - \delta^{\dot\alpha}_{\dot\gamma}\bar{y}_{\dot\alpha}y_\delta) + \delta_{\dot\alpha\dot\beta}(\sigma_m)^{\gamma\dot\delta}(-\delta^{\dot\beta}_{\dot\delta}\bar{y}_{\dot\beta}y_\gamma - \delta^{\dot\alpha}_{\dot\delta}\bar{y}_{\dot\alpha}y_\gamma)\\
&+ \delta_{\dot\alpha\dot\beta}(\sigma_m)^{\dot\gamma\dot\delta}(i\delta^{\dot\beta}_{\dot\gamma}\bar{y}_{\dot\beta}\bar{y}_{\dot\delta} + i\delta^{\dot\alpha}_{\dot\gamma}\bar{y}_{\dot\alpha}\bar{y}_{\dot\delta}) + \delta_{\dot\alpha\dot\beta}(\sigma_m)^{\dot\gamma\dot\delta}(i\delta^{\dot\beta}_{\dot\delta}\bar{y}_{\dot\beta}\bar{y}_{\dot\gamma} + i\delta^{\dot\alpha}_{\dot\delta}\bar{y}_{\dot\alpha}\bar{y}_{\dot\gamma})\Big)\\
&+ (\sigma_{nr})^{\alpha\beta}\Big(\delta_{\alpha\beta}(\sigma_m)^{\gamma\delta}(-i\delta^\beta_\gamma y_\beta y_\delta - i\delta^\alpha_\gamma y_\alpha y_\delta) + \delta_{\alpha\beta}(\sigma_m)^{\gamma\delta}(-i\delta^\beta_\delta y_\beta y_\gamma - i\delta^\alpha_\delta y_\alpha y_\gamma)\\
&+ \delta_{\alpha\beta}(\sigma_m)^{\gamma\dot\delta}(-\delta^\beta_\gamma y_\beta\bar{y}_{\dot\delta} - \delta^\alpha_\gamma y_\alpha\bar{y}_{\dot\delta}) + \delta_{\alpha\beta}(\sigma_m)^{\dot\gamma\delta}(-\delta^\beta_\delta y_\beta\bar{y}_{\dot\gamma} - \delta^\alpha_\delta y_\alpha\bar{y}_{\dot\gamma})\Big)\Big]\\
= -\frac{1}{4}\Big[&(\sigma_{nr})^{\dot\alpha\dot\beta}\Big(- \delta_{\dot\alpha\dot\beta}(\sigma_m)^{\dot\gamma\delta}\bar{y}_{\dot\gamma}y_\delta - \delta_{\dot\alpha\dot\beta}(\sigma_m)^{\gamma\dot\delta}\bar{y}_{\dot\delta}y_\gamma + i\delta_{\dot\alpha\dot\beta}(\sigma_m)^{\dot\gamma\dot\delta}\bar{y}_{\dot\gamma}\bar{y}_{\dot\delta} + i\delta_{\dot\alpha\dot\beta}(\sigma_m)^{\dot\gamma\dot\delta}\bar{y}_{\dot\delta}\bar{y}_{\dot\gamma}\Big)\\
&+ (\sigma_{nr})^{\alpha\beta}\Big(- i\delta_{\alpha\beta}(\sigma_m)^{\gamma\delta}y_\gamma y_\delta - i\delta_{\alpha\beta}(\sigma_m)^{\gamma\delta}y_\delta y_\gamma - \delta_{\alpha\beta}(\sigma_m)^{\gamma\dot\delta}y_\gamma\bar{y}_{\dot\delta} - \delta_{\alpha\beta}(\sigma_m)^{\dot\gamma\delta}y_\delta\bar{y}_{\dot\gamma}\Big)\Big]\\
= -\frac{1}{4}\Big[&- \delta_{nr}(\sigma_m)^{\dot\gamma\delta}\bar{y}_{\dot\gamma}y_\delta - \delta_{nr}(\sigma_m)^{\gamma\dot\delta}\bar{y}_{\dot\delta}y_\gamma + i\delta_{nr}(\sigma_m)^{\dot\gamma\dot\delta}\bar{y}_{\dot\gamma}\bar{y}_{\dot\delta} + i\delta_{nr}(\sigma_m)^{\dot\gamma\dot\delta}\bar{y}_{\dot\delta}\bar{y}_{\dot\gamma}\\
&- i\delta_{nr}(\sigma_m)^{\gamma\delta}y_\gamma y_\delta - i\delta_{nr}(\sigma_m)^{\gamma\delta}y_\delta y_\gamma - \delta_{nr}(\sigma_m)^{\gamma\dot\delta}y_\gamma\bar{y}_{\dot\delta} - \delta_{nr}(\sigma_m)^{\dot\gamma\delta}y_\delta\bar{y}_{\dot\gamma}\Big]
\end{aligned}$$

Now we use the following identities.

$$\delta_{nr}(\sigma_m)^{\gamma\delta} = \delta_{nr}\delta^\gamma_\epsilon\delta^\delta_\varphi(\sigma_m)^{\epsilon\varphi} = \frac{1}{2}\delta_{nr}(\sigma^n)_{\epsilon\varphi}(\sigma_n)^{\gamma\delta}(\sigma_m)^{\epsilon\varphi} = \frac{1}{2}(\sigma_r)_{\epsilon\varphi}(\sigma_n)^{\gamma\delta}(\sigma_m)^{\epsilon\varphi} = \eta_{rm}(\sigma_n)^{\gamma\delta}$$

and

$$\delta_{nr}(\sigma_m)^{\gamma\delta} = \delta_{nr}\delta^\gamma_\epsilon\delta^\delta_\varphi(\sigma_m)^{\epsilon\varphi} = \frac{1}{2}\delta_{nr}(\sigma^r)_{\epsilon\varphi}(\sigma_r)^{\gamma\delta}(\sigma_m)^{\epsilon\varphi} = -\frac{1}{2}(\sigma_n)_{\epsilon\varphi}(\sigma_r)^{\gamma\delta}(\sigma_m)^{\epsilon\varphi} = -\eta_{nm}(\sigma_r)^{\gamma\delta} \ .$$

Or equivalently, using η_{mn} for raising/lowering 4D-indices, with $\eta^{ij}\eta_{jk} = \eta_{kj}\eta^{ji} = \delta^i{}_k = \delta_k{}^i$:

$$\delta_{nr}(\sigma_m) = \delta_{sn}\delta^s_r(\sigma_m) = \delta_{sn}\eta^{sm}\eta_{mr}(\sigma_m) = \eta_n{}^m\eta_{mr}(\sigma_m) = \delta^m_n\eta_{mr}(\sigma_m) = \eta_{rm}(\sigma_n)$$
$$\delta_{nr}(\sigma_m) = \delta_{sr}\delta^s_n(\sigma_m) = \delta_{sr}\eta_{nm}\eta^{ms}(\sigma_m) = \eta_{nm}\eta^m{}_r(\sigma_m) = -\delta^m_r\eta_{nm}(\sigma_m) = -\eta_{nm}(\sigma_r) \ .$$

With this we continue with

$$[L_{\nu\rho}, K_\mu]_* \mapsto -\frac{1}{4}\Big[-\eta_{rm}(\sigma_n)^{\dot\gamma\delta}\bar y_{\dot\gamma} y_\delta - \eta_{rm}(\sigma_n)^{\gamma\dot\delta}\bar y_{\dot\delta} y_\gamma + i\eta_{rm}(\sigma_n)^{\dot\gamma\dot\delta}\bar y_{\dot\gamma}\bar y_{\dot\delta} - i\eta_{nm}(\sigma_r)^{\dot\gamma\dot\delta}\bar y_{\dot\delta}\bar y_{\dot\gamma}$$
$$- i\eta_{rm}(\sigma_n)^{\gamma\delta} y_\gamma y_\delta + i\eta_{nm}(\sigma_r)^{\gamma\delta} y_\delta y_\gamma + \eta_{nm}(\sigma_r)^{\gamma\dot\delta} y_\gamma \bar y_{\dot\delta} + \eta_{nm}(\sigma_r)^{\dot\gamma\delta} y_\delta \bar y_{\dot\gamma}\Big] \; .$$

Then we move from the dotted/undotted 4D spinors with (σ_m) to the undotted 3D spinors with (γ_μ).

$$[L_{\nu\rho}, K_\mu]_* = -\frac{i}{4}\Big(y_\gamma y_\delta - i\bar y_\gamma y_\delta - i y_\gamma \bar y_\delta - \bar y_\gamma \bar y_\delta\Big)\Big(\eta_{\rho\mu}(\gamma_\nu)^{\gamma\delta} - \eta_{\nu\mu}(\gamma_\rho)^{\gamma\delta}\Big)$$
$$= K_{\gamma\delta}\Big(\eta_{\rho\mu}(\gamma_\nu)^{\gamma\delta} - \eta_{\nu\mu}(\gamma_\rho)^{\gamma\delta}\Big) = \eta_{\rho\mu}K_\nu - \eta_{\nu\mu}K_\rho$$

6. Commutator $[L_{\nu\rho}, P_\mu]_*$

$[L_{\nu\rho}, P_\mu]_* = (\gamma_{\nu\rho})^{\alpha\beta}(\gamma_\mu)^{\gamma\delta}[L_{\alpha\beta}, P_{\gamma\delta}]_*$

With the same steps as above, just y and $\bar y$ are interchanged we get.

$$[L_{\nu\rho}, P_\mu]_* = P_{\gamma\delta}\Big(\eta_{\rho\mu}(\gamma_\nu)^{\gamma\delta} - \eta_{\nu\mu}(\gamma_\rho)^{\gamma\delta}\Big) = \eta_{rm}P_n - \eta_{nm}P_r$$

7. Commutator $[D, L_{\mu\nu}]_*$

$$[D, L_{\mu\nu}]_* = (\gamma_{\mu\nu})^{\gamma\delta}[D, L_{\gamma\delta}]_* = \frac{i}{16}(\gamma_{\mu\nu})^{\gamma\delta}\epsilon^{\alpha\beta}\Big[y_\alpha \bar y_\beta, \bar y_\gamma \bar y_\delta + y_\gamma y_\delta\Big]_*$$
$$= \frac{i}{8}(\gamma_{\mu\nu})^{\gamma\delta}\epsilon^{\alpha\beta}(\epsilon_{\beta\gamma} y_\alpha \bar y_\delta + \epsilon_{\beta\delta} y_\alpha \bar y_\gamma + \epsilon_{\alpha\gamma}\bar y_\beta y_\delta + \epsilon_{\alpha\delta}\bar y_\beta y_\gamma)$$
$$= \frac{i}{8}(\gamma_{\mu\nu})^{\gamma\delta}(-y_\sigma \bar y_\delta - y_\delta \bar y_\sigma + \bar y_\sigma y_\delta + \bar y_\delta y_\sigma) = 0$$

8. and 9. Commutator $[K_\mu, K_\nu]_*$ and $[P_\mu, P_\nu]_*$

$$[K_\mu, K_\mu]_* = (\gamma_\mu)^{\alpha\beta}(\gamma_\nu)^{\gamma\delta}[K_{\alpha\beta}, K_{\gamma\delta}]_*$$
$$= -\frac{1}{16}(\gamma_m)^{\alpha\beta}(\gamma_n)^{\gamma\delta}\Big[y_\alpha y_\beta - i\bar y_\alpha y_\beta - i y_\alpha \bar y_\beta - \bar y_\alpha \bar y_\beta, y_\gamma y_\delta - i\bar y_\gamma y_\delta - i y_\gamma \bar y_\delta - \bar y_\gamma \bar y_\delta\Big]_* =$$
$$= -\frac{1}{16}(\gamma_\mu)^{\alpha\beta}(\gamma_\nu)^{\gamma\delta}\Big(2i\epsilon_{\alpha\gamma} y_\beta y_\delta + 2i\epsilon_{\alpha\delta} y_\beta y_\gamma + (\alpha \leftrightarrow \beta)$$
$$+ 2\epsilon_{\alpha\delta} y_\beta \bar y_\gamma + 2\epsilon_{\beta\delta} y_\alpha \bar y_\gamma + (\gamma \leftrightarrow \delta)$$
$$+2\epsilon_{\beta\gamma}\bar y_\alpha y_\delta + 2\epsilon_{\beta\delta}\bar y_\alpha y_\gamma - 2i\epsilon_{\alpha\gamma} y_\beta y_\delta - 2i\epsilon_{\beta\delta}\bar y_\alpha \bar y_\gamma$$
$$-2i\epsilon_{\alpha\delta} y_\beta y_\gamma - 2i\epsilon_{\beta\gamma}\bar y_\alpha \bar y_\delta - 2\epsilon_{\alpha\gamma} y_\beta \bar y_\delta - 2\epsilon_{\alpha\delta}\bar y_\alpha y_\gamma + (\alpha \leftrightarrow \beta)$$
$$- 2\epsilon_{\alpha\gamma}\bar y_\beta y_\delta - 2\epsilon_{\beta_\gamma}\bar y_\alpha y_\delta - (\gamma \leftrightarrow \delta)$$
$$+ 2i\epsilon_{\alpha\gamma}\bar y_\beta \bar y_\delta + 2i\epsilon_{\alpha\delta}\bar y_\beta \bar y_\gamma + (\alpha \leftrightarrow \beta)\Big) = 0$$

$$[P_\mu, P_n]_* = (\gamma_\mu)^{\alpha\beta}(\gamma_\nu)^{\gamma\delta}[P_{\alpha\beta}, P_{\gamma\delta}]_*$$
$$= -\frac{1}{16}(\gamma_\mu)^{\alpha\beta}(\gamma_\nu)^{\gamma\delta}\Big[\bar y_\alpha \bar y_\beta - i y_\alpha \bar y_\beta - i\bar y_\alpha y_\beta - y_\alpha y_\beta, \bar y_\gamma \bar y_\delta - i y_\gamma \bar y_\delta - i\bar y_\gamma y_\delta - y_\gamma y_\delta\Big]_* = 0 \; .$$

C.3 O(*N*) model commutators

We now calculate the seven remaining commutators of the conformal algebra of the O(N) model what we started in chapter 4.1.

$$
\begin{aligned}
[L_{ij},P_k] &= \int d^2\vec{x}d^2\vec{y}\Big(x_i[T_{0j},T_{0k}] - x_j[T_{0i},T_{0k}]\Big)\\
&= i\int d^2\vec{x}d^2\vec{y}\bigg(x_i\pi(x)\frac{\partial}{\partial x^j}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial y^k}\phi(y) - x_i\pi(y)\frac{\partial}{\partial y^k}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial x^j}\phi(x)\\
&\qquad - x_j\pi(x)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial y^k}\phi(y) + x_j\pi(y)\frac{\partial}{\partial y^k}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial x^i}\phi(x)\bigg)\\
&= i\int d^2\vec{x}\Big(-\eta_{ij}\pi\partial_k\phi - x_i\partial_j\pi\partial_k\phi + x_i\partial_k\pi\partial_j\phi + \eta_{ij}\pi\partial_k\phi + x_j\partial_i\pi\partial_k\phi - x_j\partial_k\pi\partial_i\phi\Big)\\
&= i\int d^2\vec{x}\Big(\eta_{ij}\pi\partial_k\phi + x_i\pi\partial_k\partial_j\phi - \eta_{ik}\pi\partial_j\phi - x_i\pi\partial_j\partial_k\phi - \eta_{ji}\pi\partial_k\phi - x_j\pi\partial_k\partial_i\phi + \eta_{jk}\pi\partial_i\phi + x_j\pi\partial_i\partial_k\phi\Big)\\
&= i\bigg(\eta_{jk}\int d^3x\pi\partial_i\phi - \eta_{ik}\int d^3x\pi\partial_j\phi\bigg) = -i\bigg(\eta_{jk}\int d^3xT^0{}_i - \eta_{ik}\int d^3xT^0{}_j\bigg) = -i\Big(\eta_{jk}P_i - \eta_{ik}P_j\Big)
\end{aligned}
$$

$$
\begin{aligned}
[L_{ij},P_0] &= \int d^2\vec{x}d^2\vec{y}\Big(x_i[T_{0j},T_{00}] - x_j[T_{0i},T_{00}]\Big)\\
&= i\int d^2\vec{x}d^2\vec{y}\bigg(x_i\pi(x)\pi(y)\frac{\partial}{\partial x^j}\delta(\vec{x}-\vec{y}) - x_i\vec{\nabla}_y\phi(y)\frac{\partial}{\partial x^j}\phi(x)\vec{\nabla}_y\delta(\vec{x}-\vec{y})\\
&\qquad - x_j\pi(x)\pi(y)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}) + x_j\vec{\nabla}_y\phi(y)\frac{\partial}{\partial x^i}\phi(x)\vec{\nabla}_y\delta(\vec{x}-\vec{y})\bigg)\\
&= i\int d^2\vec{x}\Big(-\eta_{ij}\pi^2 - x_i\partial_j\pi\pi + x_i\vec{\nabla}\vec{\nabla}\phi\partial_j\phi + \eta_{ji}\pi^2 + x_j\partial_i\pi\pi - x_j\vec{\nabla}\vec{\nabla}\phi\partial_i\phi\Big) = 0\\
&= i\underbrace{\eta_{j0}}_{=0}\int d^3x\pi\partial_i\phi - i\underbrace{\eta_{i0}}_{=0}\int d^3x\pi\partial_j\phi = -i\eta_{j0}\int d^3xT^0{}_i + i\eta_{i0}\int d^3xT^0{}_j = -i\Big(\eta_{j0}P_i - \eta_{i0}P_j\Big) = 0
\end{aligned}
$$

$$
\begin{aligned}
[L_{0j},P_0] &= \int d^2\vec{x}d^2\vec{y}\Big(x_0[T_{0j},T_{00}] - x_j[T_{00},T_{00}]\Big)\\
&= i\int d^2\vec{x}d^2\vec{y}\bigg(x_0\pi(x)\pi(y)\frac{\partial}{\partial x^j}\delta(\vec{x}-\vec{y}) - x_0\vec{\nabla}_y\phi(y)\frac{\partial}{\partial x^j}\phi(x)\vec{\nabla}_y\delta(\vec{x}-\vec{y})\\
&\qquad + x_j\pi(x)\vec{\nabla}_y\phi(y)\vec{\nabla}_y\delta(\vec{x}-\vec{y}) - x_j\vec{\nabla}_x\phi(x)\pi(y)\vec{\nabla}_x\delta(\vec{x}-\vec{y})\bigg)\\
&= i\int d^2\vec{x}\Big(-\eta_{0j}\pi^2 - x_0\partial_j\pi\pi + x_0\vec{\nabla}\vec{\nabla}\phi\partial_j\phi - x_j\pi\vec{\nabla}\vec{\nabla}\phi + \vec{\nabla}x_j\vec{\nabla}\phi\pi + x_j\vec{\nabla}\vec{\nabla}\phi\pi\Big)\\
&= i\int d^2\vec{x}\Big(-x_0\partial_j\pi\pi + x_0\vec{\nabla}\vec{\nabla}\phi\partial_j\phi + \vec{\nabla}x_j\vec{\nabla}\phi\pi\Big)\\
&= i\int d^2\vec{x}\ \pi\partial_j\phi = -i\int d^2\vec{x}T^0{}_j = -iP_j = -i\Big(\eta_{j0}P_0 - \eta_{00}P_j\Big)
\end{aligned}
$$

$$
\begin{aligned}
[L_{0j}, P_k] &= \int d^2\vec{x} d^2\vec{y} \Big(x_0 [T_{0j}, T_{0k}] - x_j [T_{00}, T_{0k}] \Big) \\
&= i \int d^2\vec{x} d^2\vec{y} \Bigg(x_0 \pi(x) \frac{\partial}{\partial x^j} \delta(\vec{x}-\vec{y}) \frac{\partial}{\partial y^k} \phi(y) - x_0 \pi(y) \frac{\partial}{\partial y^k} \delta(\vec{x}-\vec{y}) \frac{\partial}{\partial x^j} \phi(x) \\
&\qquad + x_j \pi(x) \pi(y) \frac{\partial}{\partial y^k} \delta(\vec{x}-\vec{y}) - x_j \vec{\nabla}_x \phi(x) \frac{\partial}{\partial y^k} \phi(y) \vec{\nabla}_x \delta(\vec{x}-\vec{y}) \Bigg) \\
&= i \int d^2\vec{x} \Big(-\eta_{0j} \pi \partial_k \phi - x_0 \partial_j \pi \partial_k \phi + x_0 \partial_k \pi \partial_j \phi - x_j \pi \partial_k \pi + \vec{\nabla} x_j \vec{\nabla} \phi \partial_k \phi + x_j \vec{\nabla} \vec{\nabla} \phi \partial_k \phi \Big) \\
&= i \int d^2\vec{x} \Big(\eta_{0j} \pi \partial_k \phi + x_0 \pi \partial_j \partial_k \phi - \eta_{0k} \pi \partial_j \phi - x_0 \pi \partial_k \partial_j \phi + \eta_{jk} \pi^2 + x_j \partial_k \pi \pi + \vec{\nabla} x_j \vec{\nabla} \phi \partial_k \phi \\
&\qquad - \eta_{jk} \vec{\nabla} \vec{\nabla} \phi \phi - x_j \vec{\nabla} \vec{\nabla} \partial_k \phi \phi \Big) \\
&= i \int d^2\vec{x} \Big(\eta_{jk} \Big(\pi^2 + (\vec{\nabla}\phi)^2 \Big) + x_j \partial_k \pi \pi + \vec{\nabla} x_j \vec{\nabla} \phi \partial_k \phi - x_j \phi \partial_k \vec{\nabla} \vec{\nabla} \phi \Big) \\
&= i \int d^2\vec{x} \Big(\eta_{jk} \Big(\pi^2 + (\vec{\nabla}\phi)^2 \Big) + \frac{1}{2} x_j \partial_k (\pi^2) + \partial_j \phi \partial_k \phi + \vec{\nabla} x_j \vec{\nabla} \partial_k \phi \phi + x_j \partial_k \vec{\nabla} \phi \vec{\nabla} \phi \Big) \\
&= i \int d^2\vec{x} \Big(\eta_{jk} \Big(\pi^2 + (\vec{\nabla}\phi)^2 \Big) - \frac{1}{2} \eta_{jk} \pi^2 - \partial_j \partial_k \phi + \partial_j \partial_k \phi - \frac{1}{2} \eta_{jk} (\vec{\nabla}\phi)^2 \Big) \\
&= \frac{1}{2} i \eta_{jk} \int d^2\vec{x} \Big(\pi^2 + (\vec{\nabla}\phi)^2 \Big) = -i \eta_{jk} \int d^2\vec{x} T^0{}_0 = i \Big(\eta_{jk} P_0 - \underbrace{\eta_{0k}}_{=0} P_j \Big)
\end{aligned}
$$

This results in $[L_{\nu\rho}, P_\mu] = -i\Big(\eta_{\rho\mu} P_\nu - \eta_{\nu\mu} P_\rho\Big)$.

$$
\begin{aligned}
[P_i, P_j] &= \int d^2\vec{x} d^2\vec{y} [T_{0i}, T_{0j}] = \int d^2\vec{x} d^2\vec{y} \Big(\pi(x) \frac{\partial}{\partial x^i} \delta(x-y) \frac{\partial}{\partial y^j} \phi(y) - \pi(y) \frac{\partial}{\partial y^j} \delta(x-y) \frac{\partial}{\partial x^i} \phi(x) \Big) \\
&= \int d^2\vec{x} \Big(-\partial_i \pi \partial_j \phi + \partial_j \pi \partial_i \phi \Big) = \int d^2\vec{x} \Big(\pi \partial_i \partial_j \phi - \pi \partial_j \partial_i \phi \Big) \ = 0
\end{aligned}
$$

$$
[P_0, P_j] = \int d^2\vec{x} d^2\vec{y} [T_{00}, T_{0j}] = \int d^2\vec{x} (\pi \partial_j \pi - \vec{\nabla} \vec{\nabla} \partial_j \phi) = \int d^2\vec{x} \Big(\frac{1}{2} \partial_j (\pi^2) + \frac{1}{2} \partial_j ((\vec{\nabla})^2) \Big) = 0
$$

This results in $[P_\mu, P_\nu] = 0$.

$$[L_{00}, L_{0j}] = 0 = i\Big(\eta_{00}L_{0j} + \eta_{0j}L_{00} - \eta_{00}L_{0j} - \eta_{0j}L_{00}\Big)$$

$$\begin{aligned}
[L_{0i}, L_{0j}] = &\int d^2\vec{x}d^2\vec{y}\Big(x_0y_0[T_{0i}, T_{0j}] - x_iy_0[T_{00}, T_{0j}] - x_0y_j[T_{0i}, T_{00}] + x_iy_j[T_{00}, T_{00}]\Big)\\
&= i\int d^2\vec{x}d^2\vec{y}\Bigg(x_0y_0\pi(x)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial y^j}\phi(y) - x_0y_0\pi(y)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial x^i}\phi(x)\\
&\qquad + x_iy_0\pi(x)\pi(y)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y}) - x_iy_0\vec{\nabla}_x\phi(x)\frac{\partial}{\partial y^j}\phi(y)\vec{\nabla}_x\delta(\vec{x}-\vec{y})\\
&\qquad - x_0y_j\pi(x)\pi(y)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}) + x_0y_j\vec{\nabla}_y\phi(y)\frac{\partial}{\partial x^i}\phi(x)\vec{\nabla}_y\delta(\vec{x}-\vec{y})\\
&\qquad - x_iy_j\pi(x)\vec{\nabla}_y\phi(y)\vec{\nabla}_y\delta(\vec{x}-\vec{y}) + x_iy_j\vec{\nabla}_x\phi(x)\pi(y)\vec{\nabla}_x\delta(\vec{x}-\vec{y})\Bigg)\\
&= i\int d^2\vec{x}\Big(-x_0x_0\partial_i\pi\partial_j\phi + x_0x_0\partial_j\pi\partial_i\phi - x_ix_0\pi\partial_j\pi + \vec{\nabla}x_ix_0\vec{\nabla}\phi\partial_j\phi + x_ix_0\vec{\nabla}\vec{\nabla}\phi\partial_j\phi\\
&\qquad + x_0x_j\partial_i\pi\pi - x_0\vec{\nabla}x_j\vec{\nabla}\phi\partial_i\phi - x_0x_j\vec{\nabla}\vec{\nabla}\phi\partial_i\phi + x_i\vec{\nabla}x_j\pi\vec{\nabla}\phi\\
&\qquad + x_ix_j\pi\vec{\nabla}\vec{\nabla}\phi - \vec{\nabla}x_ix_j\vec{\nabla}\phi\pi - x_ix_j\vec{\nabla}\vec{\nabla}\phi\pi\Big)\\
&= i\int d^2\vec{x}\Big(-x_ix_0\pi\partial_j\pi + x_0\partial_i\phi\partial_j\phi + x_ix_0\vec{\nabla}\vec{\nabla}\phi\partial_j\phi + x_0x_j\partial_i\pi\pi - x_0\partial_j\phi\partial_i\phi\\
&\qquad - x_0x_j\vec{\nabla}\vec{\nabla}\phi\partial_i\phi + x_i\partial_j\phi\pi - x_j\partial_i\phi\pi\Big)\\
&= i\int d^2\vec{x}\Big(x_i\pi\partial_j\phi - x_j\pi\partial_i\phi\Big) = -iL_{ij} = -i\Big(\eta_{i0}L_{0j} + \eta_{0j}L_{i0} - \eta_{00}L_{ij} - \eta_{ij}L_{00}\Big)
\end{aligned}$$

$$
\begin{aligned}
[L_{ij}, L_{kl}] &= \int d^3\vec{x}d^3\vec{y}\ [x_i T_{0j} - x_j T_{0i}, y_k T_{0l} - y_l T_{0k}] \\
&= i \int d^3\vec{x}d^3\vec{y}\Big(x_i y_k [T_{0j}, T_{0l}] - x_j y_k [T_{0i}, T_{0l}] - x_i y_l [T_{0j}, T_{0k}] + x_j y_l [T_{0i}, T_{0k}]\Big) \\
&= i \int d^3\vec{x}d^3\vec{y}\Big(x_i y_k \Big(\pi(x)\frac{\partial}{\partial x^j}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial y^l}\phi(y) - \pi(y)\frac{\partial}{\partial y^l}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial x^j}\phi(x)\Big) \\
&\qquad - x_j y_k \Big(\pi(x)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial y^l}\phi(y) - \pi(y)\frac{\partial}{\partial y^l}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial x^i}\phi(x)\Big) \\
&\qquad - x_i y_l \Big(\pi(x)\frac{\partial}{\partial x^j}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial y^k}\phi(y) - \pi(y)\frac{\partial}{\partial y^k}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial x^j}\phi(x)\Big) \\
&\qquad + x_j y_l \Big(\pi(x)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial y^k}\phi(y) - \pi(y)\frac{\partial}{\partial y^k}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial x^i}\phi(x)\Big)\Big) \\
&= i \int d^3\vec{x}\Big(- \eta_{ij} x_k \pi \partial_l \phi - x_i x_k \partial_j \pi \partial_l \phi + x_i \eta_{kl} \pi \partial_j \phi + x_i x_k \partial_l \pi \partial_j \phi \\
&\qquad + \eta_{ji} x_k \pi \partial_l \phi + x_j x_k \partial_i \pi \partial_l \phi - x_j \eta_{kl} \pi \partial_i \phi - x_j x_k \partial_l \pi \partial_i \phi \\
&\qquad + \eta_{ij} x_l \pi \partial_k \phi + x_i x_l \partial_j \pi \partial_k \phi - x_i \eta_{kl} \pi \partial_j \phi - x_i x_l \partial_k \pi \partial_j \phi \\
&\qquad \underbrace{-\eta_{ji} x_l \pi \partial_k \phi}_{\text{column}=0} - x_j x_l \partial_i \pi \partial_k \phi \underbrace{+ x_j \eta_{kl} \pi \partial_i \phi}_{\text{column}=0} + x_j x_l \partial_k \pi \partial_i \phi\Big) \\
&= i \int d^3\vec{x}\Big(\eta_{ij} x_k \pi \partial_l \phi + x_i \eta_{kj} \pi \partial_l \phi - \eta_{il} x_k \pi \partial_j \phi - x_i \eta_{kl} \pi \partial_j \phi \\
&\qquad - \eta_{ji} x_k \pi \partial_l \phi - x_j \eta_{ki} \pi \partial_l \phi + \eta_{jl} x_k \pi \partial_i \phi + x_j \eta_{kl} \pi \partial_i \phi \\
&\qquad - \eta_{ij} x_l \pi \partial_k \phi - x_i \eta_{lj} \pi \partial_k \phi + \eta_{ik} x_l \pi \partial_j \phi + x_i \eta_{lk} \pi \partial_j \phi \\
&\qquad \underbrace{+\eta_{ji} x_l \pi \partial_k \phi}_{\text{column}=0} + x_j \eta_{li} \pi \partial_k \phi - \eta_{jk} x_l \pi \partial_i \phi \underbrace{- x_j \eta_{lk} \pi \partial_i \phi}_{\text{column}=0}\Big) \\
&= -i \int d^3\vec{x}\Big(\eta_{il}(x_j {T^0}_k - x_k {T^0}_j) + \eta_{jk}(x_i {T^0}_l - x_l {T^0}_i) - \eta_{ik}(x_j {T^0}_l - x_l {T^0}_j) - \eta_{jl}(x_i {T^0}_k - x_k {T^0}_i)\Big) \\
&= -i\Big(\eta_{il} L_{jk} + \eta_{jk} L_{il} - \eta_{ik} L_{jl} - \eta_{jl} L_{ik}\Big)
\end{aligned}
$$

This results in $[L_{\mu\nu}, L_{\rho\sigma}] = -i\Big(\eta_{\mu\sigma} L_{\nu\rho} + \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}\Big)$.

$$
\begin{aligned}
[D,K_j] &= \int d^2\vec{x}d^2\vec{y}\Big[x^0{T^0}_0 + x^i{T^0}_i - \frac{1}{2}\phi(x)\pi(x), y^2{T^0}_j - 2y_jy^0{T^0}_0 - 2y_jy^k{T^0}_k + y_j\phi(y)\pi(y) + \frac{1}{2}\delta_j^0\phi(y)^2\Big] \\
&= \int d^2\vec{x}d^2\vec{y}\Big(x^0y^2[T_{00},T_{0j}] + x^iy^2[T_{0i},T_{0j}] - 2x^0y_jy^0[T_{00},T_{00}] \\
&\quad - 2x^0y_jy^k[T_{00},T_{0k}] - 2x^iy_jy^0[T_{0i},T_{00}] - 2x^iy_jy^k[T_{0i},T_{0k}] \\
&\quad - \frac{1}{2}y^2[\phi(x)\pi(x),{T^0}_j] + y_jy^0[\phi(x)\pi(x),{T^0}_0] + y_jy^i[\phi(x)\pi(x),{T^0}_i] \\
&\quad - \frac{1}{2}y_j[\phi(x)\pi(x),\phi(y)\pi(y)] - \frac{1}{4}\delta_j^0[\phi(x)\pi(x),\phi(y)^2] + x^0y_j[{T^0}_0,\phi(y)\pi(y)] \\
&\quad + x^iy_j[{T^0}_i,\phi(y)\pi(y)] + \frac{1}{2}x^0\delta_j^0[{T^0}_0,\phi(y)^2] + \frac{1}{2}x^i\delta_j^0[{T^0}_i,\phi(y)^2]\Big) \\
&= i\int d^2\vec{x}d^2\vec{y}\Big(- x^0y^2\pi(x)\pi(y)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y}) + x^0y^2\vec{\nabla}_x\phi(x)\frac{\partial}{\partial y^j}\phi(y)\vec{\nabla}_x\delta(\vec{x}-\vec{y}) \\
&\quad + x^iy^2\pi(x)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial y^j}\phi(y) - x^iy^2\pi(y)\frac{\partial}{\partial y^j}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial x^i}\phi(x) \\
&\quad + 2x^0y_jy^0\pi(x)\vec{\nabla}_y\phi(y)\vec{\nabla}_y\delta(\vec{x}-\vec{y}) - 2x^0y_jy^0\vec{\nabla}_x\phi(x)\pi(y)\vec{\nabla}_x\delta(\vec{x}-\vec{y}) \\
&\quad + 2x^0y_jy^k\pi(x)\pi(y)\frac{\partial}{\partial y^k}\delta(\vec{x}-\vec{y}) - 2x^0y_jy^k\vec{\nabla}_x\phi(x)\frac{\partial}{\partial y^k}\phi(y)\vec{\nabla}_x\delta(\vec{x}-\vec{y}) \\
&\quad - 2x^iy_jy^0\pi(x)\pi(y)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y}) + 2x^iy_jy^0\vec{\nabla}_y\phi(y)\frac{\partial}{\partial x^i}\phi(x)\vec{\nabla}_y\delta(\vec{x}-\vec{y}) \\
&\quad - 2x^iy_jy^k\pi(x)\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial y^k}\phi(y) + 2x^iy_jy^k\pi(y)\frac{\partial}{\partial y^k}\delta(\vec{x}-\vec{y})\frac{\partial}{\partial x^i}\phi(x)\Big) \\
&\quad + \int d^2\vec{x}d^2\vec{y}\Big(\frac{y^2}{2}[\phi(x)\pi(x),\pi(y)\partial_j\phi(y)] - \frac{y_j}{2}y^0[\phi(x)\pi(x),\pi(y)^2] - \frac{y_j}{2}y^0[\phi(x)\pi(x),\vec{\nabla}\phi(y)^2] \\
&\quad - y_jy^i[\phi(x)\pi(x),\pi(y)\partial_i\phi(y)] - \frac{1}{4}\delta_j^0[\phi(x)\pi(x),\phi(y)^2] \\
&\quad - \frac{x^0}{2}y_j[\pi(x)^2,\phi(y)\pi(y)] - \frac{x^0}{2}y_j[\vec{\nabla}\phi(x)^2,\phi(y)\pi(y)] - x^iy_j[\pi(x)\partial_i\phi(x),\phi(y)\pi(y)] \\
&\quad - \frac{x^0}{4}\delta_j^0[\pi(x)^2,\phi(y)^2] - \frac{x^0}{4}\delta_j^0[\vec{\nabla}\phi(x)^2,\phi(y)^2] - \frac{1}{2}x^i\delta_j^0[\pi(x)\partial_i\phi(x),\phi(y)^2]\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= i\int d^2\vec{x}\Big(2x^0x_j\pi^2 + x^0x^2\pi\partial_j\pi - x^0x^2\vec{\nabla}\vec{\nabla}\phi\partial_j\phi \\
&\quad - 2x^2\pi\partial_j\phi - x^ix^2\partial_i\pi\partial_j\phi + 2x^ix_j\pi\partial_i\phi + x^ix^2\partial_j\pi\partial_i\phi \\
&\quad - 2x^0\vec{\nabla}x_jx^0\pi\vec{\nabla}\phi - 2x^0x_jx^0\pi\vec{\nabla}\vec{\nabla}\phi + 2x^0x_jx^0\vec{\nabla}\vec{\nabla}\phi\pi \\
&\quad - 2x^0x_j\pi^2 - 4x^0x_j\pi^2 - 2x^0x_jx^k\pi\partial_k\pi + 2x^0x_jx^k\vec{\nabla}\vec{\nabla}\phi\partial_k\phi \\
&\quad + 4x_jx^0\pi^2 + 2x^ix_jx^0\partial_i\pi\pi - 2x^i\vec{\nabla}x_jx^0\vec{\nabla}\phi\partial_i\phi - 2x^ix_jx^0\vec{\nabla}\vec{\nabla}\phi\partial_i\phi \\
&\quad + 4x_jx^k\pi\partial_k\phi + 2x^ix_jx^k\partial_i\pi\partial_k\phi - 2x^ix_j\pi\partial_i\phi - 4x^ix_j\pi\partial_i\phi - 2x^ix_jx^k\partial_k\pi\partial_i\phi\Big) \\
&\quad + \int d^2\vec{x}d^2\vec{y}\Big(\frac{y^2}{2}[\phi(x),\pi(y)]\pi(x)\partial_j\phi(y) + \frac{y^2}{2}[\pi(x),\partial_j\phi(y)]\phi(x)\pi(y) - y_jx^0[\phi(x),\pi(y)]\pi(x)\pi(y) \\
&\quad - y_jx^0[\pi(x),\vec{\nabla}\phi(y)]\phi(x)\vec{\nabla}\phi(y) - y_jy^i[\phi(x),\pi(y)]\pi(x)\partial_i\phi(y) - y_jy^i[\pi(x),\partial_i\phi(y)]\phi(x)\pi(y) \\
&\quad - \frac{1}{2}\delta^0_j[\pi(x),\phi(y)]\phi(x)\phi(y) - x^0y_j[\pi(x),\phi(y)]\pi(x)\pi(y) - x^0y_j[\vec{\nabla}\phi(x),\pi(y)]\vec{\nabla}\phi(x)\phi(y) \\
&\quad - x^iy_j[\pi(x),\phi(y)]\partial_i\phi(x)\pi(y) - x^iy_j[\partial_i\phi(x),\pi(y)]\pi(x)\phi(y) \\
&\quad - x^0\delta^0_j[\pi(x),\phi(y)]\pi(x)\phi(y) - x^i\delta^0_j[\pi(x),\phi(y)]\partial_i\phi(x)\phi(y)\Big) \\
&= i\int d^2\vec{x}\Big(- 4x^0x_j\pi^2 + x^0x^i\partial_i\phi\partial_j\phi + x^0x_i\partial^i\phi\partial_j\phi - x_jx^0(\vec{\nabla}\phi)^2 + x^2\pi\partial_j\phi - 2x^ix_j\pi\partial_i\phi \\
&\quad - 2x^0x_jx^k\pi\partial_k\pi - 2x^ix^0\partial_j\phi\partial_i\phi + \frac{1}{2}x^2\pi\partial_j\phi + x_j\phi\pi + \frac{1}{2}x^2\phi\partial_j\phi - x_jx^0\pi^2 - x^0\phi\partial_j\phi - x_jx^0\phi\vec{\nabla}\vec{\nabla}\phi \\
&\quad - x_jx^i\pi\partial_i\phi - x_j\phi\pi - 2x_j\phi\pi - x_jx^i\phi\partial_i\pi + \frac{1}{2}\delta^0_j\phi^2 + x^0x_j\pi^2 + x^0x_j\vec{\nabla}\vec{\nabla}\phi\phi + x^ix_j\partial_i\phi\pi \\
&\quad + 2x_j\pi\phi + x^ix_j\partial_i\pi\phi + x^0\delta^0_j\pi\phi + x^i\delta^0_j\partial_i\phi\phi\Big) \\
&= i\int d^2\vec{x}\Big(- x^0x_j\pi^2 - x^0x_j(\vec{\nabla}\phi)^2 + x^2\pi\partial_j\phi - 2x^ix_j\pi\partial_i\phi - x_j\phi\pi + \frac{1}{2}\delta^0_j\phi^2 + x^i\delta^0_j\partial_i\phi\phi\Big) \\
&= -i\int d^2\vec{x}\Big(- x^2\pi\partial_j\phi + x_jx^0\pi^2 + x_jx^0(\vec{\nabla}\phi)^2 + 2x_jx^i\pi\partial_i\phi + x_j\phi\pi + \frac{1}{2}\delta^0_j\phi^2\Big) = -iK_j
\end{aligned}
$$

$$
\begin{aligned}
[D, K_0] &= \int d^2\vec{x} d^2\vec{y} \Big[x^0 {T^0}_0 + x^i {T^0}_i - \frac{1}{2}\phi(x)\pi(x), y^2 {T^0}_0 - 2y_0 y^0 {T^0}_0 - 2y_0 y^k {T^0}_k + y_0\phi(y)\pi(y) + \frac{1}{2}\delta^0_0 \phi(y)^2 \Big] \\
&= \int d^2\vec{x} d^2\vec{y} \Big(x^0 y^2 [T_{00}, T_{00}] + x^i y^2 [T_{0i}, T_{00}] - 2x^0 y_0 y^0 [T_{00}, T_{00}] \\
&\quad - 2x^0 y_0 y^k [T_{00}, T_{0k}] - 2x^i y_0 y^0 [T_{0i}, T_{00}] - 2x^i y_0 y^k [T_{0i}, T_{0k}] \\
&\quad - \frac{1}{2} y^2 [\phi(x)\pi(x), {T^0}_0] + y_0 y^0 [\phi(x)\pi(x), {T^0}_0] + y_0 y^i [\phi(x)\pi(x), {T^0}_i] \\
&\quad - \frac{1}{2} y_0 [\phi(x)\pi(x), \phi(y)\pi(y)] - \frac{1}{4}\delta^0_0 [\phi(x)\pi(x), \phi(y)^2] + x^0 y_0 [{T^0}_0, \phi(y)\pi(y)] \\
&\quad + x^i y_0 [{T^0}_i, \phi(y)\pi(y)] + \frac{1}{2} x^0 [{T^0}_0, \phi(y)^2] + \frac{1}{2} x^i [{T^0}_i, \phi(y)^2] \Big) \\
&= i \int d^2\vec{x} d^2\vec{y} \Big(- x^0 y^2 \pi(x) \vec{\nabla}_y \phi(y) \vec{\nabla}_y \delta(\vec{x} - \vec{y}) + x^0 y^2 \vec{\nabla}_x \phi(x) \pi(y) \vec{\nabla}_x \delta(\vec{x} - \vec{y}) \\
&\quad + x^i y^2 \pi(x)\pi(y) \frac{\partial}{\partial x^i} \delta(\vec{x} - \vec{y}) - x^i y^2 \vec{\nabla}_y \phi(y) \frac{\partial}{\partial x^i} \phi(x) \vec{\nabla}_y \delta(\vec{x} - \vec{y}) \\
&\quad + 2x^0 y_0 y^0 \pi(x) \vec{\nabla}_y \phi(y) \vec{\nabla}_y \delta(\vec{x} - \vec{y}) - 2x^0 y_0 y^0 \vec{\nabla}_x \phi(x) \pi(y) \vec{\nabla}_x \delta(\vec{x} - \vec{y}) \\
&\quad + 2x^0 y_0 y^k \pi(x)\pi(y) \frac{\partial}{\partial y^k} \delta(\vec{x} - \vec{y}) - 2x^0 y_0 y^k \vec{\nabla}_x \phi(x) \frac{\partial}{\partial y^k} \phi(y) \vec{\nabla}_x \delta(\vec{x} - \vec{y}) \\
&\quad - 2x^i y_0 y^0 \pi(x)\pi(y) \frac{\partial}{\partial x^i} \delta(\vec{x} - \vec{y}) + 2x^i y_0 y^0 \vec{\nabla}_y \phi(y) \frac{\partial}{\partial x^i} \phi(x) \vec{\nabla}_y \delta(\vec{x} - \vec{y}) \\
&\quad - 2x^i y_0 y^k \pi(x) \frac{\partial}{\partial x^i} \delta(\vec{x} - \vec{y}) \frac{\partial}{\partial y^k} \phi(y) + 2x^i y_0 y^k \pi(y) \frac{\partial}{\partial y^k} \delta(\vec{x} - \vec{y}) \frac{\partial}{\partial x^i} \phi(x) \Big) \\
&\quad + \int d^2\vec{x} d^2\vec{y} \Big(\frac{y^2}{4} [\phi(x)\pi(x), \pi(y)^2] + \frac{y^2}{4} y2 [\phi(x)\pi(x), (\vec{\nabla}\phi(y))^2] - \frac{y_0}{2} y^0 [\phi(x)\pi(x), \pi(y)^2] \\
&\quad - \frac{y_0}{2} y^0 [\phi(x)\pi(x), (\vec{\nabla}\phi(y))^2] - y_0 y^i [\phi(x)\pi(x), \pi(y)\partial_i\phi(y)] - \frac{1}{4} [\phi(x)\pi(x), \phi(y)^2] \\
&\quad - \frac{x^0}{2} y_0 [\pi(x)^2, \phi(y)\pi(y)] - \frac{x^0}{2} y_0 [(\vec{\nabla}\phi(x))^2, \phi(y)\pi(y)] - x^i y_0 [\pi(x)\partial_i\phi(x), \phi(y)\pi(y)] \\
&\quad - \frac{x^0}{4} [\pi(x)^2, \phi(y)^2] - \frac{x^0}{4} [(\vec{\nabla}\phi(x))^2, \phi(y)^2] - \frac{x^i}{2} [\pi(x)\partial_i\phi(x), \phi(y)^2] \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= i\int d^2\vec{x}\Big(x^0\vec{\nabla}(x^2)\pi\vec{\nabla}\phi + x^0x^2\pi\vec{\nabla}\vec{\nabla}\phi - x^0x^2\vec{\nabla}\vec{\nabla}\phi\pi \\
&\quad - 2x^2\pi^2 - x^ix^2\partial_i\pi\pi + x^i\vec{\nabla}(x^2)\vec{\nabla}\phi\partial_i\phi + x^ix^2\vec{\nabla}\vec{\nabla}\phi\partial_i\phi \\
&\quad - 2x^0x_0x^0\pi\vec{\nabla}\vec{\nabla}\phi + 2x^0x_0x^0\vec{\nabla}\vec{\nabla}\phi\pi \\
&\quad - 4x^0x_0\pi^2 - 2x^0x_0x^k\pi\partial_k\pi + 2x^0x_0x^k\vec{\nabla}\vec{\nabla}\phi\partial_k\phi \\
&\quad + 4x_0x^0\pi^2 + 2x^ix_0x^0\partial_i\pi\pi - 2x^ix_0x^0\vec{\nabla}\vec{\nabla}\phi\partial_i\phi \\
&\quad + 4x_0x^k\pi\partial_k\phi + 2x^ix_0x^k\partial_i\pi\partial_k\phi - 4x^ix_0\pi\partial_i\phi - 2x^ix_0x^k\partial_k\pi\partial_i\phi\Big) \\
&\quad + \int d^2\vec{x}d^2\vec{y}\Big(\frac{y^2}{2}[\phi(x),\pi(y)]\pi(x)\pi(y) + \frac{y^2}{2}[\pi(x),\vec{\nabla}\phi(y)]\phi(x)\vec{\nabla}\phi(y) - y^0y_0[\phi(x),\pi(y)]\pi(x)\pi(y) \\
&\quad - y_0y^0[\pi(x),\vec{\nabla}\phi(y)]\phi(x)\vec{\nabla}\phi(y) - y_0y^i[\phi(x),\pi(y)]\pi(x)\partial_i\phi(y) - y_0y^i[\pi(x),\partial_i\phi(y)]\phi(x)\pi(y) \\
&\quad - \frac{1}{2}[\pi(x),\phi(y)]\phi(x)\phi(y) - x^0y_0[\pi(x),\phi(y)]\pi(x)\pi(y) - x^0y_0[\vec{\nabla}\phi(x),\pi(y)]\vec{\nabla}\phi(x)\phi(y) \\
&\quad - x^iy_0[\pi(x),\phi(y)]\partial_i\phi(x)\pi(y) - x^iy_0[\partial_i\phi(x),\pi(y)]\pi(x)\phi(y) \\
&\quad - x^0[\pi(x),\phi(y)]\pi(x)\phi(y) - x^i[\pi(x),\phi(y)]\partial_i\phi(x)\phi(y)\Big) \\
&= i\int d^2\vec{x}\Big(2x^0x^i\pi\partial_i\phi - x_0x^0\pi^2 - x^2(\vec{\nabla}\phi)^2 + x^2(\vec{\nabla}\phi)^2 + x^ix_i(\vec{\nabla}\phi)^2 + \frac{x^2}{2}\pi^2 + \frac{1}{2}\vec{\nabla}(x^2)\phi\vec{\nabla}\phi \\
&\quad + \frac{x^2}{2}\phi\vec{\nabla}\vec{\nabla}\phi - x_0x^0\pi^2 - x_0x^0\phi\vec{\nabla}\vec{\nabla}\phi - x_ox^i\pi\partial_i\phi - 2x_0\phi\pi - x_0x^i\phi\partial_i\pi + \frac{1}{2}\phi^2 + x^0x_0\pi^2 \\
&\quad + x^0x_0\vec{\nabla}\vec{\nabla}\phi\phi + x^ix_0\partial_i\phi\pi + 2x_0\pi\phi + x^ix_0\partial_i\pi\phi + x^0\pi\phi + x^i\partial_i\phi\phi\Big) \\
&= i\int d^2\vec{x}\Big(-2x_0x^i\pi\partial_i\phi - x_0x^0\pi^2 + \frac{x^2}{2}(\vec{\nabla}\phi)^2 - x_0x^0(\vec{\nabla}\phi)^2 + \frac{x^2}{2}\pi^2 + \frac{1}{2}\phi^2 + x^0\pi\phi - \phi^2\Big) \\
&= -i\int d^2\vec{x}\Big(-\frac{x^2}{2}\pi^2 - \frac{x^2}{2}(\vec{\nabla}\phi)^2 + x_0x^0\pi^2 + x_0x^0(\vec{\nabla}\phi)^2 + 2x_0x^k\pi\partial_k\phi + x_0\phi\pi + \frac{1}{2}\phi^2\Big) \\
&= -iK_0
\end{aligned}
$$

This results in $[D, K_\mu] = -iK_\mu$.

With similar calculations we get

$$[D, L_{\nu\rho}] = 0 \quad , \quad [K_\mu, K_\nu] = 0 \quad , \quad [L_{\nu\rho}, K_\mu] = -i\Big(\eta_{\rho\mu}K_\nu - \eta_{\nu\mu}K_\rho\Big) \; .$$

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Dedication and acknowlegdement

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Selbstständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Jena, den 19. April 2016

Seitens des Verfassers bestehen keine Einwände, die vorliegende Masterarbeit für die öffentliche Nutzung in der Thüringer Universitäts- und Landesbibliothek zur Verfügung zu stellen.

Jena, den 19. April 2016